RECONSTRUCTION, QUANTIFICATION, AND VISUALIZATION OF FOREST CANOPY BASED ON 3D TRIANGULATIONS OF AIRBORNE LASER SCANNING POINT DATA

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Commission III, WG III/4

KEY WORDS: Forest inventory, Tree allometry, Light Detection and Ranging (LiDAR), Delaunay triangulation, Alpha shape

ABSTRACT:

Reconstruction of three-dimensional (3D) forest canopy is described and quantified using airborne laser scanning (ALS) data with densities of 0.6–0.8 points m^{-2} and field measurements aggregated at resolutions of 400–900 m^2. The reconstruction was based on computational geometry, topological connectivity, and numerical optimization. More precisely, triangulations and their filtrations, i.e. ordered sets of simplices belonging to the triangulations, based on the point data were analyzed. Triangulating the ALS point data corresponds to subdividing the underlying space of the points into weighted simplicial complexes with weights quantifying the (empty) space delimited by the points. Reconstructing the canopy volume populated by biomass will thus likely require filtering to exclude that volume from canopy voids. The approaches applied for this purpose were (i) to optimize the degree of filtration with respect to the field measurements, and (ii) to predict this degree by means of analyzing the persistent homology of the obtained triangulations, which is applied for the first time for vegetation point clouds. When derived from optimized filtrations, the total tetrahedral volume had a high degree of determination (R^2) with the stem volume considered, both alone (R^2=0.65) and together with other predictors (R^2=0.78). When derived by analyzing the topological persistence of the point data and without any field input, the R^2 were lower, but the predictions still showed a correlation with the field-measured stem volumes. Finally, producing realistic visualizations of a forested landscape using the persistent homology approach is demonstrated.

1. INTRODUCTION

Airborne laser scanning (ALS) has become a routinely operated technique for obtaining wall-to-wall data for vast geographical areas (e.g., White et al., 2013; Maltamo et al., 2014). From a practical point of view, currently the most interesting ALS data acquisitions are those carried out by several European countries for ground elevation modeling (e.g. Nord-Larsen and Riis-Nielsen, 2010; Dalponte et al., 2011; Gomez-Gutierrez et al., 2011; Bohlin et al., 2012; Villikka et al., 2012). Due to their regional to national coverage, these data provide a great potential also for other applications such as environmental mapping, monitoring and modelling.

The sparse point densities (<1 pulses m^{-2}) of these data typically limit the analyses to the use of area-based approaches of the distributions of height values (e.g., Næsset, 2002). Notably, such analyses employ only one of the three dimensions available in the data and ignore the (horizontal) between-point information. Triangulating point data, i.e. subdividing the underlying space of the points into simplices, and subsequent filtering of these triangulations (e.g., Delfinado and Edelsbrunner, 1995) has been proposed as an alternative means of representing the 3D canopy surface (Vauhkonen et al. 2014).

Compared to approaches like constructing pixel- or voxel-based canopy height models from the data, a practical benefit of using a triangulation-based approach is that instead of needing to specify an artificial pixel or voxel resolution, the triangulations are entirely based on the properties of the point data. Applying filtrations, one can adjust the level of detail in the triangulated point cloud and thus account for canopy gaps and detailed properties existing in the data with the parameterization of the filtration (see Figure 1; Section 2.3.). Finally, the use of triangulations and filtrations provides means of constructing genuine 3D visualizations of forested landscapes instead of just representing counts of height values within pixel or voxel cells.

The filtering step is particularly needed to exclude the volume populated by canopy biomass from that representing empty space. A particularly interesting approach for the filtering is the concept of 3D alpha shapes (Edelsbrunner and Mücke, 1994), or α-shapes, in which a predefined parameter α is used as a size-criterion to determine the level of detail in the obtained triangulation. An important step is thus the selection of α, for which the selected value should enable separation of void spaces from those populated by canopy biomass. For this purpose, Vauhkonen et al. (2014) proposed classifying the simplices into either canopy biomass or voids based on numerical optimization with field-measured biomass. This is laborious due to the requirement of field data.

The main idea here is to test whether the appropriate level of detail of the filtrations could be determined by analyzing the inherent algebraic properties of the triangulations such as simplicial homomorphism and topological persistence

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(Zomorodian, 2005). Earlier, Martynov (2008), computing persistent homology groups at different spatial resolutions and assuming that important features and structures persisted over a wide range of scales, de-noised geometric models of power line towers reconstructed from the ALS point data. Other corresponding approaches to the point data depicting man-made objects can be pointed out. To the best of the author's knowledge, however, the application described in Section 2.4 is the first ever analysis of persistent homology of vegetation point clouds.

The purpose of this study was thus to test the potential of analyzing the persistent homology of the triangulated point data to generate reliable 3D models of forest canopy. The resulting triangulations, filtrations and canopy volumes were evaluated against the field measurements of stem volume, and compared with predictions based on using an optimized canopy volume and the conventional height and density metrics for the same purpose. Finally, the approach is demonstrated for a fully automated visualization of a forested landscape.

2. METHODS

2.1 Material

2.1.1 Study area: The study area is a typical boreal managed forest area in eastern Finland (62°31'N, 30°10'E) with Scots pine (*Pinus sylvestris* L.) as the dominant tree species. It represents 73% of the volume, Norway spruce (*Picea abies* [L.] Karst.) 16% of the volume, and deciduous trees altogether about 11% of the volume.

2.1.2 Field measurements: Altogether 79 field plots were measured on May – June 2010. The plots were placed subjectively, attempting to record the species and size variation over the area. Sample plot size varied between 20×20 and 30×30 m according to stand development class. The tree measurements included species, DBH and height (H) for each tree. Stem volumes (V) were calculated using the DBH and H estimates in the species-specific volume equations described by Laasasenaho (1982).

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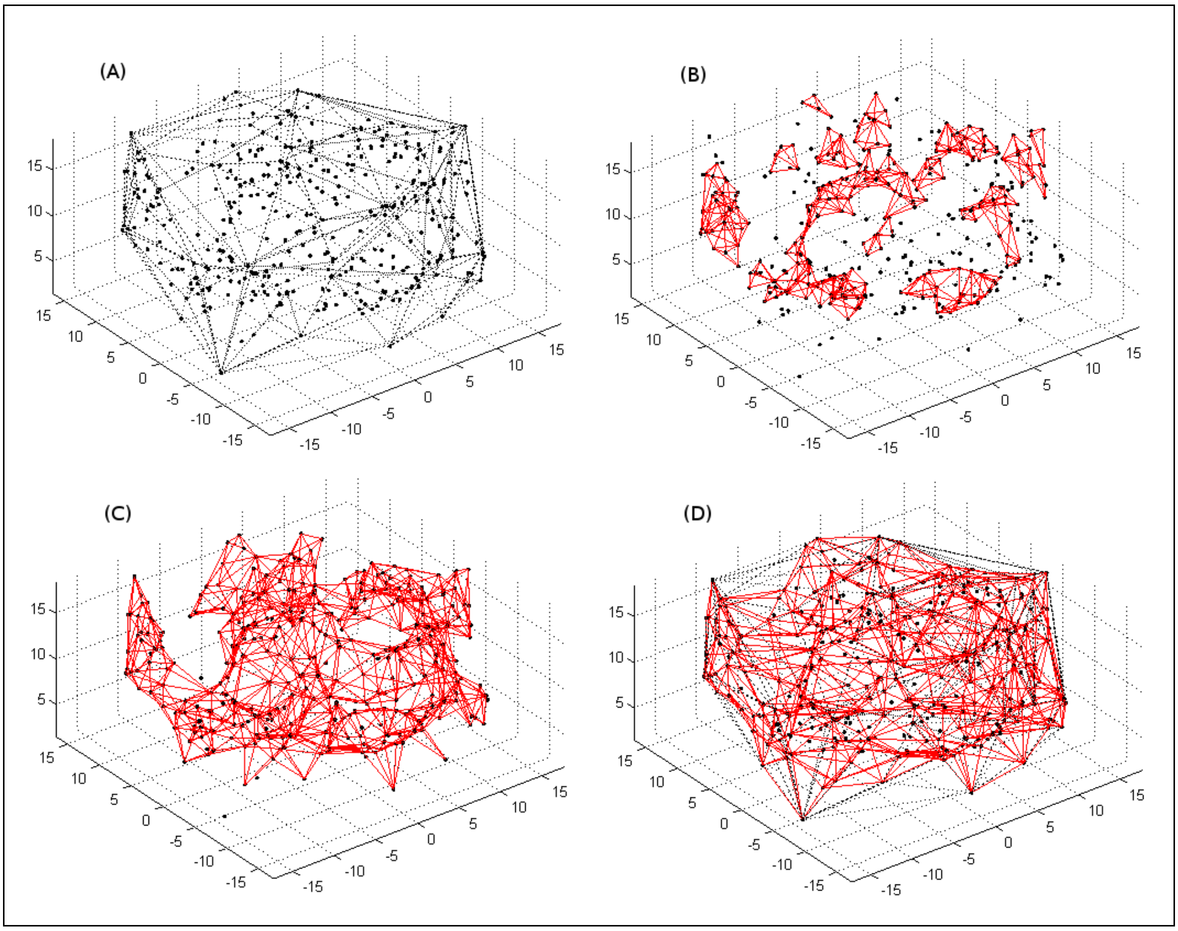

Figure 1. Triangulation and filtration of an example set of 3D points illustrated with symbols corresponding to Section 2.3. Subplot (A) shows the extreme values of the filtration: the initial set of points (i=0, F[0]=0, $C_0=\emptyset$, and $V_{C,0}=0$ m^3) and the convex hull of the point data (i=s=2081, F[2081]=∞, C_{2081}=all, and $V_{C,2081}$=11343.6 m^3) denoted by dashed lines. The red lines in subplots B, C, and D correspond to subcomplexes with F[214]=3.5m and $V_{C,214}$=206.7 m^3, F[746]=7.2m and $V_{C,746}$=1379.1 m^3, and F[1477]=18.7 m and $V_{C,1477}$=5206.5 m^3, respectively.

2.1.3 ALS data: The ALS data used in the study were acquired in two separate acquisitions. All numerical analyses were carried out with data collected on 26 June 2009 using Optech ALTM Gemini system from a flying height of 2000 m and with a nominal density of 0.65 points m^{-2}. A separate data set collected on 30 April 2012 with Leica ALS 60 system from a flying height of 2350 m and a density of 0.79 points m^{-2} was further utilized for visualization. Both the data sets were pre-processed in a similar way, i.e. the ground surface was detected and classified, and the echo heights normalized with respect to this surface following a standard TerraScan approach. Only the first echoes (i.e. 'only' and 'first-of-many' of multiple echoes recorded per pulse) extracted employing a ground threshold of 1 m were included in the analyses

2.2 An overview of the methods

The main aim was to model the 3D canopy by means of triangulating the ALS point data, i.e. subdividing the underlying space of the points into simplices, which results in weighted simplicial complexes (e.g. Edelsbrunner, 2011) with weights quantifying the (empty) space delimited by the points. Reconstructing the canopy volume populated by biomass will thus likely require filtering to exclude that volume from canopy voids (Figure 1). For this purpose, we considered the following steps, which are explained in detail in the subsequent sections.

1. Define filtering rules to be able to adjust the level of detail in the triangulation based on the ALS data of separate plots (Section 2.3.).

2. Define the degree of filtering such that the resulting filtration corresponded the reference forest measurements as closely as possible. For this step, we considered two alternatives: (i) derive the degree of filtering by analyzing the topological persistence of the point data (Section 2.4.), and (ii) optimize the degree of filtering with the reference basal area observed from the plots used in step 1 (Section 2.5.).

3. Evaluate and visualize the predictions obtained (Section 2.6.).

2.3 Deriving triangulations and their filtrations

The terminology and notation in the following are adapted from Delfinado and Edelsbrunner (1995), to which the reader is referred for additional details. A triangulation T of a finite set of points S is a subdivision of the convex hull of S into simplices according to the rules of the particular triangulation (see, e.g., Preparata and Shamos, 1985). T is a simplicial complex that is subsettable to a finite number of subcomplexes C_i, i=1, 2, …, s, where i indexes the position and s is the number of different subcomplexes in a list of potential filter values F defining which simplices belong to the subcomplex (see the next paragraph). When F is ordered (here ascendingly), the sequence C_i constitutes a filtration, in which each complex is a sub-complex of its successor, and $V_{C,i}$ is the volume of the underlying space of subcomplex C_i according to this filtration.

The point data of each individual plot were triangulated and filtered separately. The filtration corresponded to the family of 3D α-shapes (Edelsbrunner and Mücke, 1994), each of which consist of those simplices of T, which have an empty circumscribing sphere with a squared radius smaller than α. Although an α-shape is defined for every non-negative real α, there are only finitely many different α-complexes (see Delfinado and Edelsbrunner, 1995, p. 782), and F was thus an ascending sequence of those α-values which altered the α-shape. All computations were based on Delaunay triangulations and implemented with C++ and an open-source Computational Geometry Algorithms Library (Pion and Teillaud, 2013; Da and Yvinec, 2013).

2.4 Persistent homology

During the filtration of the data (Figure 1), i.e. by increasing the considered filtration parameter α, simplices of a dimension d join together with other d-dimensional simplices to form structures with a higher dimensionality. Particularly, when the value of α allows two points (d=0) to connect, an edge with d=1 between these two points is formed. The edges further join to form facets (d=2) and tetrahedra (d=3), the latter being the highest dimension considered here. Each simplex can thus be assigned by a value describing its persistence in the structure formed. This index of persistence was in this case obtained as the absolute difference between the index values of α causing the birth and death of simplices of the given dimension.

Following the logic of the previous paragraph, a birth and death diagram (Figure 2) was computed for indices with d=1 and d=2. Only these dimensions need to be considered, since those with d=0 or d=3 do not born or die, respectively, in this process. In the diagram, the most interesting observations are those off-diagonal: the simplices are interpret to cause the most fundamental changes to the triangulated structure, typically at the lowest values of α, while diagonality of the observations indicates stability (persistence) of those features. The value for the α parameter was thus decided separately for each plot as the maximum off-diagonal α at the death of both d=1 and d=2.

The reasoning behind this approach is the attempt to discover such a value for the parameter α, beyond which the changes to the filtration were irrelevant in that those did not affect the primary structures of the obtained filtration. An example birth and death diagram corresponding to the point data and triangulation in Figure is given by Figure 2, while the mathematical formalism of this approach is presented in much more detail by Edelsbrunner et al. (2002).

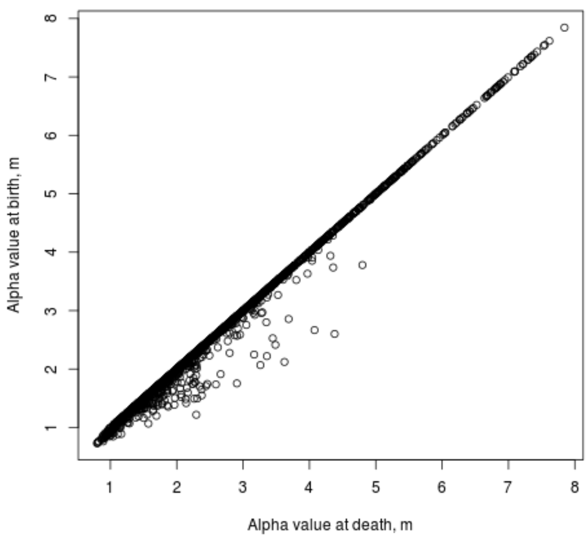

Figure 2. An example birth and death diagram, corresponding to the point data and triangulation of Figure 1. The selected value of α was approximately 5.8 m for this plot.

2.5 Filtering the triangulations by an optimization with respect to forest biomass in the training plots

To optimize the filtrations, the studied set of field plots was considered jointly. For convenience, let $V_{C,i,p}$ in the following text denote the volume of the underlying space of a subcomplex C_i of the ALS point data of plot *p*. The optimization was based on minimizing the residual errors of a linear regression model

$$G_p = \beta_1 + \beta_2 V_{C,i,p} + \varepsilon_p, \quad (1)$$

where G_p was the reference basal area, β_1 and β_2 the model parameters, and ε_p the residual error of plot *p*. Although the ALS point data could be expected to represent canopy biomass (branches and foliage) rather than that originating from the stems, *G* is the most typical forest biophysical property measured in forest inventories with strong allometric links to other forest variables (see further notes in Section 5). Therefore we optimized Eq. 1 with respect to this variable.

The initial values of β_1 and β_2 were obtained by fitting the model to $V_{C,i,p}$ with *i* selected randomly for each plot. In the optimization, $V_{C,i,q}$ of plot *q* which produced the largest residual error was altered to $V_{C,i+10,q}$ in successive rounds of 500 iterations, until either the error, determined as 1 – the obtained coefficient of determination (R^2), decreased below 0.0009 or did not change during the last round of iterations. The whole procedure was repeated 100 times, which resulted in range of filtration parameters and thus canopy volumes that were in a quasi-optimal linear relationship with forest biomass in at least one out of the 100 solutions. The per-plot mean value of these ranges was used in the subsequent analyses.

2.6 Evaluation, performance measures, and visualization

The numerical evaluation of our approach was based on assessing the correspondence of the total tetrahedral volume of the filtrations with the plot-level stem volume measured in the field. This relationship was quantified by fitting an allometric power equation (White and Gould, 1965):

$$y_p = b\, V_{C,i,p}{}^k + \varepsilon_p, \quad (2)$$

where y_p was the reference stem volume, $V_{C,i,p}$ the tetrahedral volume of the underlying space of the selected subcomplex $C_{i,p}$, *b* and *k* the model parameters, and ε_p the residual error of plot *p*. The model fitting was carried out with the nls function of R (R Core Team, 2012).

The allometric power equation corresponding to Eq. 2 was also fit between the reference stem volumes and the most common ALS-based predictor variables (cf. Næsset 2002). The latter were the mean and standard deviation of the height values, the proportion of echoes above 2 m, and the 5th, 10th, 20th, …, 90th, and 95th percentiles and the corresponding proportional densities of the ALS-based canopy height distribution, calculated according to Korhonen et al. (2008, pp. 502–503).

The correspondence of the estimates with the reference data was evaluated by graphical assessments and in terms of R^2, root mean squared error (RMSE) and mean difference (bias). The latter two were calculated using the standard equations presented by Vauhkonen et al. (2014), for example.

To demonstrate the potential of the proposed approach for 3D visualizations of forest canopy, the ALS point data of a forested area of approximately 1000 × 1200 m was divided on a grid with a cell size of 20 × 20 m. Each cell of this grid was separately triangulated and analyzed for its persistent homology. Therefore, the only input for the visualization of the forest was the ALS point data, while the filtration parameter α corresponded to the value obtained following Section 2.4. The locations of the lakes and roads in the area were extracted from a topographic database maintained by the National Land Survey of Finland and inserted manually to the image. The image of the entire forest area was finally drawn with graphical functions of Matlab.

3. RESULTS

Approximately 50% of the optimized solutions (Eq. 1) yielded $R^2 \geq 0.985$ and the average R^2 was 0.93. The mean values of the 100 iterations of optimizing Eq. 1 produced a nearly 1:1 relationship with the plot-level basal area used in the optimization ($R^2 \approx 1.0$). Due to the allometric relationships between the basal area and other forest attributes, the tetrahedral volume was in a good correspondence also with the main evaluated attribute, i.e. the total stem volume. For example, the optimization produced an R^2 of 0.65 between these attributes.

The tetrahedral volume derived by analyzing the persistent homology of the triangulations was not as accurate as the reference canopy volume produced by optimization (Table 1, Figure 3). However, noting that no allometric information on the forest attributes was used in the derivation (unlike in the case of the optimized one), the metric deduced from the properties of the point data is somewhat related to stem volume as well.

When inserted jointly to Eq. 2, the conventional height and density metrics showed a better degree of determination between the predicted and reference values than the volumetric features (Table 1). However, inserting the tetrahedral volume obtained by analyzing the persistent homology of the point data improved the model despite the low degree of determination of this metric in the first place.

Finally, despite the low degree of determination with the field measured stem volume, the positions of the canopy elements derived with the persistent homology approach corresponded well to those of the field-measured tree stems (Figure 4), which showed a considerable potential in producing realistic visualizations of a forested landscape (Figure 5). When looked more closely and with respect of an aerial or satellite image, the canopy volume produced by the approach seems to realistically convey information of the real canopy and existence of canopy gaps. Furthermore, although there are discontinuities between the 20 × 20 m cells, these do not seem to prevent a landscape-level visualization with the approach proposed.

Predictors	R^2	RMSE, m^3/ha	bias, m^3/ha
CV_O	0.65	44.3	-3.5
CV_P	0.17	66.7	6.8
H	0.65	43.5	-7.5
D	0.16	67.0	1.1
H + D	0.78	34.0	4.4
H + D + CV_P	0.78	33.9	5.4

Table 1. Accuracies of predicting the plot-level stem volume with Eq. 2 fit with the given predictors. CV_O – the tetrahedral volume of the optimized filtrations, CV_P – the tetrahedral volume derived by persistent homology, H – the most correlated height feature (90th percentile), D – the most correlated density feature (proportion of echoes above 1 m height). The average stem volume was 197.6 m^3/ha.

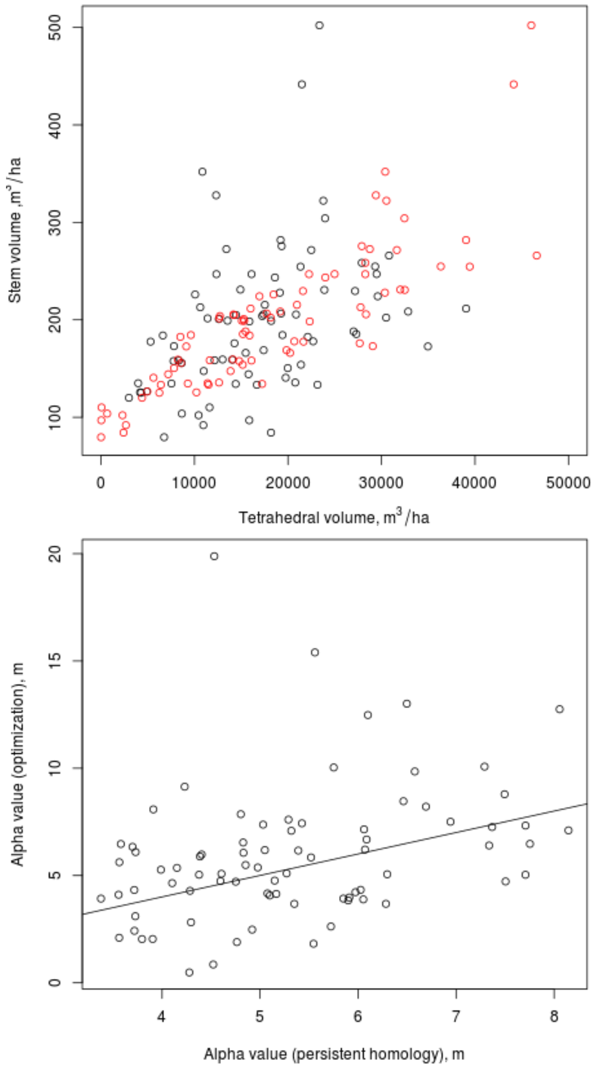

Figure 3. Above: the relationship of the canopy volume (persistent homology – black circles, optimization – red circles) with the plot-level stem volume. Below: the correspondence of the α values obtained by the two approaches.

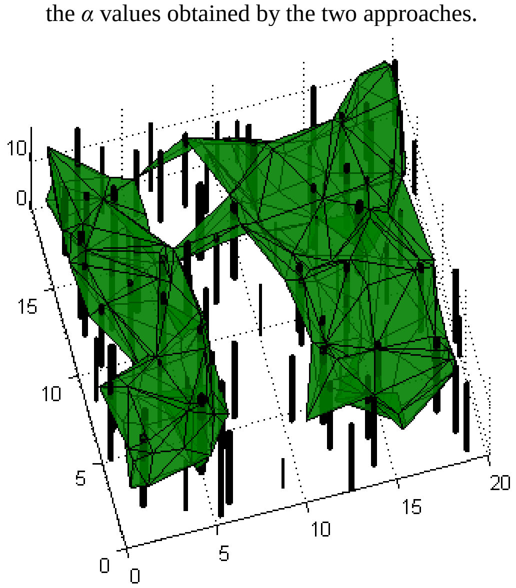

Figure 4. Canopy elements produced by the persistent homology approach vs. tree stems observed within the area of an example plot.

4. DISCUSSION

The results of this study verified the observations made in another test site by Vauhkonen et al. (2014), particularly in that the triangulation-based approach to analyze the sparse-density vegetation point clouds is an interesting alternative to the conventionally used height and density metrics. Further, since the low resolution of the data likely disables the possibility to use individual tree detection, using the triangulation-based approach more information on the within-plot variation is obtained.

The triangulations are entirely based on the properties of the point data, which can be extracted from practically available data with sparse point densities (see also Vauhkonen et al. 2012; Maltamo et al. 2010). Applying filtrations, one can adjust the level of detail in the triangulated point cloud and thus account for canopy gaps and detailed properties existing in the data, which is controlled by filtration parameter α. Regarding the latter, the results of this study can be divided to two distinct parts: those obtainable in the presence of both field reference and ALS data, and those obtainable for a prediction grid. We consider both of the results important, for simulations and predictions, respectively.

The approach here was to triangulate the 3D point data and to filter the triangulations for empty space. When the degree of the filtration was determined in an optimization, the derived tetrahedral volumes were more closely related to the stem biomass that was considered for evaluation. Earlier (Vauhkonen et al. 2014), the filtration parameter obtained via optimization was regressed with the height and density metrics, which produced a reasonably accurate result, but was found feasible only in the presence of an adequate field data set, most preferably representing field observations of canopy biomass due to the link with the canopy volume.

The idea of optimization (Eq. 2) is to determine the filtering degree such that a quasi-optimal relationship is formed between the considered forest attribute and an attribute characterizing the filtrations. Vauhkonen et al. (2014) used the modeled canopy- and above-ground biomasses for this purpose, but found the resulting canopy volumes to also have a strong allometric relationship with all other forest attributes considered. As a variate of the tree diameter, G, on the other hand, is directly measured from the trees and also the most typical forest attribute obtained in aggregate-level forest inventories. The use of G was thus justified by the practical applicability of the proposed approach, but also by the unknown form of the “real” allometric relationship between the canopy volume and forest attributes of interest (see Discussion by Vauhkonen et al. 2014). Whether the relationship formed by other variables was better justified, both G and the tetrahedral volume currently used for characterizing the simplicial complexes could be substituted in Eq. 2 by these variables.

Further, unlike in previous studies, the approaches used here to determine the filtration parameter were (i) to optimize the degree of filtration with respect to field measured basal area (G), and (ii) to predict this degree by means of analyzing the persistent homology of the obtained triangulations. The selection of the parameter α turned out to be highly sensitive toward the obtained results. When the degree of the filtration was determined in an optimization, the filtrations and their total tetrahedral volumes were more closely related to the biomass attributes than in the latter case. Another alternative would be to

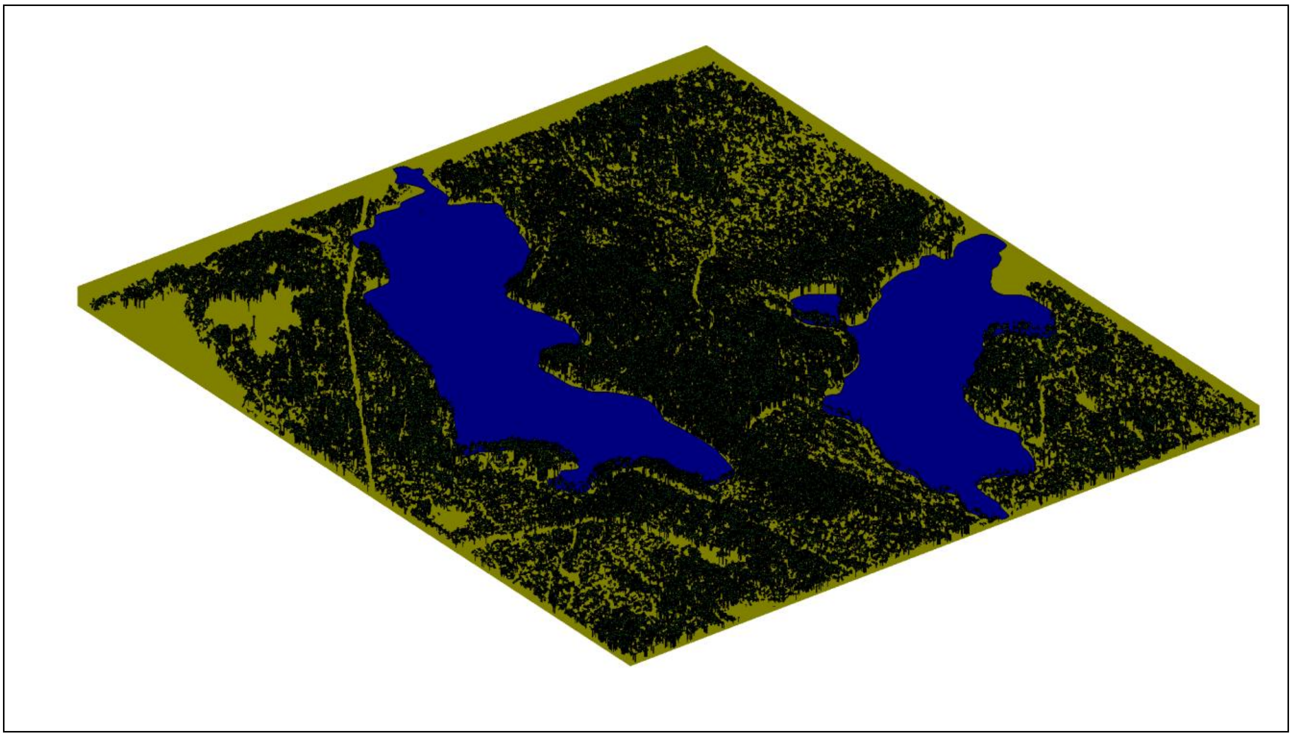

Figure 5. A visualization of a forested landscape produced by the persistent homology approach proposed.

determine the most feasible population-specific fixed value like Vauhkonen et al. (2012), who evaluated a range of α's, but rather than selecting one particular value, they used metrics quantifying the difference of the shape obtained with the range of α's to the convex hull of the point data, which corresponds to α-shapes with $\alpha \rightarrow \infty$. An exploratory analysis carried out prior to this study indicated lower degrees of determination could be obtained by a fixed value than by a plot-specific value.

The persistent homology approach was tested to replace the need for the field data by analyzing the inherent topological properties of the point data. A simple approach based on the birth and death diagrams of the various simplices (Edelsbrunner et al., 2002) turned out to produce reasonably good information. It is of author's belief that persistent homology of vegetation point clouds should be inspected more carefully after this initial test. For example, one could derive the Euler characteristic and the Betti numbers from the filtrations (Robins, 2002), and use them in a similar manner than the birth and death diagram was used. One could further attempt to quantify the obtained filtration by other geometric features than total volumes such as edge lengths or facet areas. The approach has a profound mathematical formalism on topological connectivity rather than the alternative approaches based on the tetrahedral volume. Except for the optimization, the implementations here were based on efficient data structures (Pion and Teillaud, 2013; Da and Yvinec, 2013) and are computationally feasible for practical applications.

Although the evaluation with respect to the stem volume is a natural starting point for forest inventories, it should be noted that the triangulations and the filtrations could likely be related to other forest biophysical properties than the biomass-based variables considered. For example, the spatial distribution of the tetrahedra could potentially be related to the spatial pattern of the trees (see Figure 4), the prediction of which has been found difficult based on both the area- and tree-level analyses of ALS data (cf. Packalén et al., 2013). Another application could be the visualization of a forest landscape (see Figure 5), which is proposed to benefit e.g. forest management planning (e.g. Mendoza et al. 2006). The results promote the usability of the α-shapes (Edelsbrunner and Mücke, 1994) for this purpose in that instead of aiming to produce the current status of the forest, one could adjust the value of α to present the magnitude of cuttings and evaluate these effects to the landscape produced. This type of problem setting will be considered in our future research.

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Submitted November 2014
Revised February 2015