NONZONAL EXPRESSIONS OF GAUSS- KRÜGER PROJECTION IN POLAR REGIONS

Zhongmei Li, Shaofeng Bian*, Qiang Liu, Houpu Li, Cheng Chen, Yanfeng Hu

Dept. of Navigation, Naval University of Engineering, Wuhan, 430033 China - lizhongmei126@126.com

Commission Ⅳ, WG Ⅳ/2

KEY WORDS: Gauss-krüger projection, transverse Mercator projection, polar surveying, reference frame, conformal colatitude, computer algebra system

ABSTRACT:

With conformal colatitude introduced, based on the mathematical relationship between exponential and logarithmic functions by complex numbers, strict equation of complex conformal colatitude is derived, and then theoretically strict nonzonal expressions of Gauss projection in polar regions are carried out. By means of the computer algebra system, correctness of these expressions is verified, and sketches of Gauss-krüger projection without bandwidth restriction in polar regions are charted. In the Arctic or Antarctic region, graticule of nonzonal Gauss projection complies with people's reading habit and reflects real ground-object distribution. Achievements in this paper could perfect mathematical basis of Gauss projection and provide reference frame for polar surveying and photogrammetry.

1. INTRODUCTION

Polar regions have increasingly been the international focus in recent decades. It is of great significance for polar navigation and scientific investigation to select the suitable projection method. As one common conformal projection, the transverse Mercator (TM) or Gauss-Krüger projection is frequently used for charting topographic map (e.g., Lauf 1983; Snyder 1987; Yang 2000). Series expansions of meridian length in Krüger (1912) is the basis of the most common way for calculation of Gauss coordinates. In the last century, scholars have carried on extensive researches about the projection. Lee (1976) and Dozier (1980) carried out formulae of UTM coordinates by means of elliptic functions. Based on complex numbers, Bowring (1990) gave one improved solution for TM projection. With respect to Laplace-Beltrami and Korn-Lichtenstein equations, conformal coordinates of type UTM or Gauss-Krüger were carried out directly in Grafarend (2006). Additionally, Bermejo (2009) derived simple and highly accurate formulas of TM coordinates from longitude and isometric latitude, and compared truncation errors in different orders by using the program Maple and Matlab. Karney (2011) extended Krüger's series to 8th order, constructed high-precision test set based on Lee (1976) and discussed properties of the exact mapping far from the central meridian. Obviously, researches on TM or Gauss-Krüger projection have already obtained brilliant achievement.

In the development history of Gauss-krüger projection theories, formulae above have different features, for example, real power series expansions of longitude difference l are often limited in a narrow strip (e.g. $l \in [-3.5^\circ, +3.5^\circ]$ in UTM projection; $l \in [-3^\circ, +3^\circ]$ or $l \in [-1.5^\circ, +1.5^\circ]$ in Gauss-Krüger projection). Expressions by complex numbers, eliminating zoning restrictions, are difficult to be used in polar regions with the singularity of isometric latitude. Karney (2011) improved Lee's formulae, and provided an accuracy of 9 nm over the entire ellipsoid, bur not gave formulae that can entirely express the Arctic or Antarctic region. In attempts on the nonzonal formulae of Gauss projection in polar regions, Bian (2014) used a near-spherical assumption to derive complex colatitude, which would have an influence on the strictness of his formulae. Given these, in order to perfect the mathematical foundations of Gauss-krüger projection specialized in polar regions, by introducing the relationship between conformal colatitude and isometric latitude, an improvement measure will

* Corresponding author

be shown in this paper.

2. EXPRESSIONS OF GAUSS COORDINATES IN NONPOLAR REGION

According to Bian (2012), based on meridian arc length expansion about conformal latitude, non-iterative expressions of Gauss projection are written as in this form

$$\begin{cases} \boldsymbol{\varphi} = \arcsin\left(\tanh \mathbf{w}\right) \\ \mathbf{z} = x + iy = a(\alpha_0 \boldsymbol{\varphi} + \alpha_2 \sin 2\boldsymbol{\varphi} + \alpha_4 \sin 4\boldsymbol{\varphi} + \\ \quad \alpha_6 \sin 6\boldsymbol{\varphi} + \alpha_8 \sin 8\boldsymbol{\varphi} + \alpha_{10} \sin 10\boldsymbol{\varphi} \cdots) \end{cases} \tag{1}$$

where a indicates semi-major axis of the earth ellipsoid. Coefficients $\alpha_0, \alpha_2 \cdots \alpha_{10}$ expanded to e^{10} are carried out by computer algebra system Mathematica. With its strong power in symbolic operation, coefficients expanded to e^{20} or even e^{40} can be easily gotten in a similar way. And finally all the coefficients could be simplified as series summations of earth ellipsoid eccentricity. As our target in this paper is to improve original formulae of Gauss projection for polar using, we place great importance on the improvement measures, and do not discuss the expansion coefficients whether they are expanded enough to a high precision or not. Here we take the first eccentricity e for example, coefficients $\alpha_0 \cdots \alpha_{10}$ expanded to e^{10} are listed in Eq. (2).

$$\begin{cases} \alpha_0 = 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 - \frac{441}{65536}e^{10} \\ \alpha_2 = \frac{1}{8}e^2 - \frac{1}{96}e^4 - \frac{9}{1024}e^6 - \frac{901}{184320}e^8 - \frac{16381}{5898240}e^{10} \\ \alpha_4 = \frac{13}{768}e^4 + \frac{17}{5120}e^6 - \frac{311}{737280}e^8 - \frac{18931}{20643840}e^{10} \\ \alpha_6 = \frac{61}{15360}e^6 + \frac{899}{430080}e^8 + \frac{14977}{27525120}e^{10} \\ \alpha_8 = \frac{49561}{41287680}e^8 + \frac{175087}{165150720}e^{10} \\ \alpha_{10} = \frac{34729}{82575360}e^{10} \end{cases} \tag{2}$$

Additionally, in Eq. (1) $\boldsymbol{\varphi}$ and $\mathbf{w}$ indicates complex conformal latitude and complex isometric latitude respectively, and

$$\mathbf{w} = q + il \tag{3}$$

where l indicates geodetic longitude, q means isometric latitude and is a function of geodetic latitude B

$$\begin{aligned} q(B) &= \operatorname{arctanh}\left(\sin B\right) - e \operatorname{arctanh}\left(e \sin B\right) \\ &= \ln \sqrt{\frac{1+\sin B}{1-\sin B}\left(\frac{1-e\sin B}{1+e\sin B}\right)^e} \end{aligned} \tag{4}$$

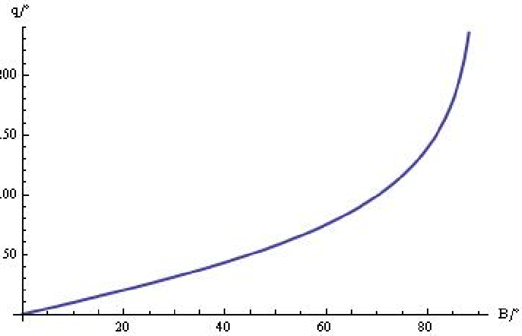

Figure 1. Sketch of isometric latitude q with geodetic latitude $B \in [0^\circ, 90^\circ)$

Taking $\operatorname{arctanh}\left(e\sin\left(-B\right)\right) = -\operatorname{arctanh}\left(e\sin B\right)$ and $\operatorname{arctanh}\left(\sin\left(-B\right)\right) = -\operatorname{arctanh}\left(\sin B\right)$ into account, $q(-B) = -q(B)$ is gotten from Eq. (4), meaning that isometric latitude q is an odd function of geodetic latitude B. Trend of isometric latitude q with geodetic latitude B ranging from 0° to 90° is charted in Figure 1.

As shown in Fig. 1, isometric latitude q increases with geodetic latitude B ranging from 0° to 90°. In consideration of isometric latitude q 's odevity, q becomes an infinitely large quantity as geodetic latitude B approaches to $\pm 90^\circ$, which brings about singularity in expressions of complex isometric latitude $\mathbf{w}$ in Eq. (3) as well as complex conformal latitude $\boldsymbol{\varphi}$ in Eq. (1), and then makes expressions of Gauss coordinates in Eq. (1) difficult to be used in polar zones.

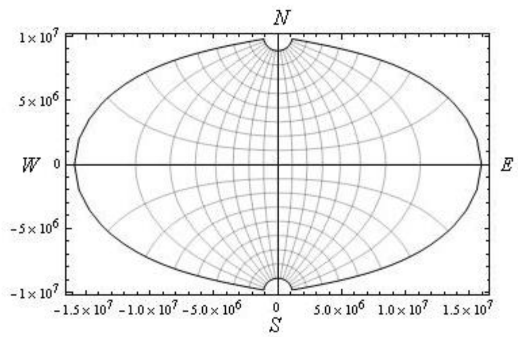

Figure 2. Sketch of Gauss projection in nonpolar region

Moreover, as $\tanh \mathbf{w} = \tanh\left(q + il\right)$ in Eq. (1) contains $\tanh\left(il\right) = i \tan l$, not suit for the situation where geodetic longitude l approaches to 90°, expressions of Gauss coordinates in nonpolar region Eq. (1) only can be used in the zone $D=\{(B,l):|l|<90^\circ,|B|<90^\circ\}$. By means of computer algebra system Mathematica, sketch of Gauss projection in nonpolar region is drawn in Figure 2.

3. NONSINGULAR EXPRESSIONS OF GAUSS COORDINATES IN POLAR REGIONS

In order to carry out the expressions of Gauss projection that can be used in polar regions, Eq. (1) must be transformed equivalently. As Eq. (1) is derived from meridian arc length expansion

$$\begin{aligned} X = a\,(&\alpha_0\varphi+\alpha_2\sin 2\varphi+\alpha_4\sin 4\varphi+ \\ &\alpha_6\sin 6\varphi+\alpha_8\sin 8\varphi+\alpha_{10}\sin 10\varphi+\cdots) \end{aligned} \tag{5}$$

where X indicates meridian arc length, φ indicates conformal latitude, and coefficients $\alpha_0\cdots\alpha_{10}$ are the same as Eq. (1). According to Xiong (1988), conformal latitude φ is a function of geodetic latitude B

$$\varphi = 2\arctan\left[\tan\left(\frac{\pi}{4}+\frac{B}{2}\right)\left(\frac{1-e\sin B}{1+e\sin B}\right)^{e/2}\right]-\frac{\pi}{2} \tag{6}$$

When the geodetic latitude B is on the northern hemisphere, the conformal latitude φ is a positive value. Otherwise, it is a negative value.

3.1 Nonsingular Expressions of Gauss Projection in Complex Form

To eliminate the singularity of conformal latitude φ when geodetic latitude B approaches to 90° in Eqs. (5)—(6), conformal colatitude θ is introduced, and values

$$\theta = \pi/2-\varphi \tag{7}$$

Afterwards, inserting Eq. (7) into Eq. (5), equivalent expression of meridian length X can be written with conformal colatitude θ .

$$\begin{aligned} X = a\,(&-\alpha_0\theta+\alpha_2\sin 2\theta-\alpha_4\sin 4\theta+\alpha_6\sin 6\theta- \\ &\alpha_8\sin 8\theta+\alpha_{10}\sin 10\theta-\cdots)+a\alpha_0\pi/2 \end{aligned} \tag{8}$$

Obviously, singularity of Eq. (8) depends on the singularity of the unique variable θ . To judge the singularity of θ, inserting Eq. (6) into Eq. (7), equation of conformal colatitude θ is gotten.

$$\begin{aligned} \theta &= \pi-2\arctan\left[\tan\left(\frac{\pi}{4}+\frac{B}{2}\right)\left(\frac{1-e\sin B}{1+e\sin B}\right)^{e/2}\right] \\ &= \pi-2\arctan\left[\exp(q)\right]=2\operatorname{arccot}\left[\exp(q)\right] \\ &= 2\arctan\left[\exp(-q)\right] \end{aligned} \tag{9}$$

Taking Eq. (4) into account, based on the relationship between exponential and logarithmic functions $\exp(\ln x)\equiv x$, Eq. (10) is gotten.

$$\begin{aligned} \exp(-q) &= \exp\left[-\ln\sqrt{\frac{1+\sin B}{1-\sin B}\left(\frac{1-e\sin B}{1+e\sin B}\right)^{e}}\right] \\ &= \sqrt{\frac{1-\sin B}{1+\sin B}\left(\frac{1+e\sin B}{1-e\sin B}\right)^{e}} \end{aligned} \tag{10}$$

Inserting Eq. (10) into Eq. (9), we can find when $B=90^\circ$, $\theta=0$. Neither θ nor Eq. (8) is singular in north pole. As nonpolar solution of Gauss projection is obtained by developing the relationship between meridian length X and isometric latitude q from real to complex number field, expressions used in polar regions can be achieved similarly.

Firstly, based on the definition of complex function, $\mathbf{w}=q+il$ replaces q in Eq. (9) to realize the extension of conformal colatitude θ, and then the complex conformal colatitude $\boldsymbol{\theta}$ is derived.

$$\boldsymbol{\theta} = 2\arctan\left[\exp\left(-(q+il)\right)\right] \tag{11}$$

Secondly, replacing θ in Eq. (8) by $\boldsymbol{\theta}$, and turning meridian length X in Eq. (8) into complex coordinates $\mathbf{z}=x+iy$, where the real part x indicates Gauss ordinate and the imaginary part y indicates abscissa, yields to expressions of Gauss projection. For convenient polar charting, moving zero point of the expressions from the equator to the north pole, ordinate is reduced by $1/4$ meridian arc length ($a\alpha_0\pi/2$) and abscissa remains unchanged. For clear presentation, abscissa and ordinate after translation are still expressed with y and x . Omitting the derivation, nonsingular expression of Gauss coordinates by complex numbers in Arctic region is carried out.

$$\mathbf{z} = x + i\,y = a\big(-\alpha_0\boldsymbol{\theta} + \alpha_2 \sin 2\boldsymbol{\theta} - \alpha_4 \sin 4\boldsymbol{\theta} + \alpha_6 \sin 6\boldsymbol{\theta} - \alpha_8 \sin 8\boldsymbol{\theta} + \alpha_{10} \sin 10\boldsymbol{\theta} - \cdots\big) \tag{12}$$

When longitude $l = 0^\circ$, abscissa $y = 0$, ordinate x equals to meridian arc length integrating from the pole. Equality of the central meridian arc is guaranteed. Besides, transformations above are all elementary operations between complex functions, which are monodrome and analytic functions in the principle value, keep conformal in the whole transformation processes. It is verified that Eq. (12) satisfy Cauchy—Riemann equations, so conformality of Gauss projection is guaranteed.

By now, nonzonal solution of Gauss projection that can be used in Arctic region have been finished.

3.2 Nonsingular Expressions of Gauss Projection in Real Form

As complex conformal colatitude $\boldsymbol{\theta}$ is a complex variable, to separate $\boldsymbol{\theta}$ into real and imaginary parts $\boldsymbol{\theta} = u + iv$, equations $\exp(q + il) = \exp(q)(\cos l + i \sin l)$ and $q = \operatorname{arctanh}(\sin\varphi)$ are introduced. Based on the relationship between complex function and arctangent function, Eq. (12) is transformed equivalently.

$$\begin{aligned}
\boldsymbol{\theta} &= 2\arctan\big[\exp\big(-(q + il)\big)\big] \\
&= -i \ln \frac{1 + i\exp(-q)(\cos l - i \sin l)}{1 - i\exp(-q)(\cos l - i \sin l)} \\
&= -i \ln\left(\frac{\sinh q}{\cosh q - \sin l} + i \frac{\cos l}{\cosh q - \sin l}\right) \\
&= \arctan(\operatorname{csch} q \cos l) - i \operatorname{arctanh}(\operatorname{sech} q \sin l) \\
&= \arctan(\tan\theta \cos l) - i \operatorname{arctanh}(\sin\theta \sin l)
\end{aligned} \tag{13}$$

After transformed, complex conformal colatitude $\boldsymbol{\theta}$ is devided into real and imaginary part, they equals

$$\begin{cases} u = \arctan(\tan\theta\cos l) \\ v = -\operatorname{arctanh}(\sin\theta\sin l) \end{cases} \tag{14}$$

Obviously, when $P(B, l)$ approaches to the north pole, $\theta \to 0$, range of l reaches to $[-180^\circ, 180^\circ]$, and $\boldsymbol{\theta}$ has a specific value and not singular at certain point $P(B, l)$ on the northern hemisphere.

Taking relationships $\sin(u + iv) = \sin u \cos iv + \cos u \sin iv$, $\sin iv = i \sinh v$ and $\cos iv = \cosh v$ into account, Eq. (11) is separated into real and imaginary parts, and Eq. (15) is gotten.

$$\begin{cases} x = -a\alpha_0 u + a\sum\limits_{n=1}^{\infty}(-1)^{n-1}\alpha_{2n}\sin(2nu)\cosh(2nv) \\ y = -a\alpha_0 v + a\sum\limits_{n=1}^{\infty}(-1)^{n-1}\alpha_{2n}\cos(2nu)\sinh(2nv) \end{cases} \tag{15}$$

By now, nonzonal expressions of Gauss projection in real form suit for the Arctic region have been carried out.

Actually, taking full advantage of earth's symmetry, it is no need to derive extra expressions of Gauss coordinates for the Antarctic region. As the southern hemisphere is symmetrical to the northern hemisphere, viewing the south pole as new north pole, replacing $P(B, l)$ on southern hemisphere by $P(-B, l)$ and inserting it into expressions used in Arctic region, sketch of the Antarctic region in the same perspective as the Arctic region is gotten. Sketches of Gauss projection in polar regions are drawn in Figures 3 and 4.

As shown in Figures 3 and 4, the Arctic and Antarctic regions are completely displayed based on nonzonal expressions above. Through translating the origin of Gauss coordinates from the equator to the pole, Gauss ordinates are negative when longitude difference $|l| < 90^\circ$, while ordinates are positive when longitude difference $|l| > 90^\circ$. In polar circles, the meridians $l = 0^\circ, \pm 90^\circ, \pm 180^\circ$ are shown as straight lines after projected, and are the symmetry axes of Gauss coordinates.

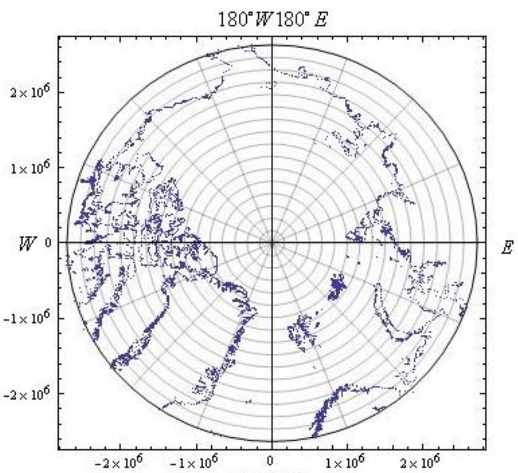

Figure 3. Sketch of Gauss projection in Arctic region based on nonzonal expressions

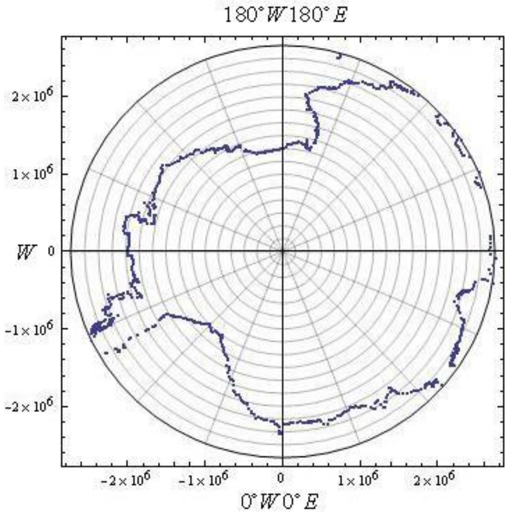

Figure 4. Sketch of Gauss projection in Antarctic region based on nonzonal expressions

4. CONCLUSIONS

With equations of conformal colatitude and isometric latitude introduced, based on relationship between complex exponential and logarithmic functions, nonzonal expressions of Gauss projection in polar regions are carried out. Conclusions are drawn as below.

(1) With isometric latitude singular in the pole, traditional expressions of Gauss projection can not be used in polar regions. Through translating the origin of traditional Gauss projection by $1/4$ meridian length from the equator to the pole, theoretically strict expressions that can be used in polar regions are carried out. These are of great significance for perfecting mathematical system of Gauss projection.

(2) Compared with traditional Gauss projection, nonzonal formulae derived in this paper are fit for the whole polar regions without bandwidth restriction, and could provide reference frame for polar surveying and photogrammetry.

(3) Though parallel circles and the other meridians are not projected to straight lines like Mercator projection, graticule of Gauss projection in the Arctic or Antarctic region could still comply with our reading habit, even reflect real ground-object distribution better and make the polar regions absolutely clear at a glance.

ACKNOWLEDGES

This work is financially supported by National 973 Plan Program (No. 2012CB719902) and National Natural Science Foundation of China (No. 41471387, 41571441).

REFERENCES

Bermejo, M., Otero, J., 2009. Simple and highly accurate formulas for the computation of Transverse Mercator coordinates from longitude and isometric latitude. *Journal of Geodesy*, 83, pp. 1-12.

Bian, S. F., Li, H. P., 2012. Mathematical analysis in cartography by means of computer algebra system. In: *Cartography-A Tool for Spatial Analysis*, InTech, Croatia, Chapter 1, pp. 1-24.

Bian, S. F., Li, Z. M., Li, H. P., 2014. The non-singular formula of Gauss Projection in polar regions by complex numbers. *Acta Geodaetica et Cartographica Sinica*, 43(4), pp. 348-352.

Bowring, B. R., 1990. The transverse Mercator projection—a solution by complex numbers. *Survey Review*, 30(237), pp. 325-342.

Dozier, J., 1980. Improved algorithm for calculation of UTM coordinates and geodetic coordinates. NOAA Technical Report NESS 81, National Oceanic and Atmospheric Administration, Washington.

Grafarend, E., Krumm, F., 2006. *Map Projections*. Springer, Berlin.

Karney, C. F. F., 2011. Transverse Mercator with an accuracy of a few nanometers. *Journal of Geodesy*, 85(8), pp. 475-485.

Krüger, L., 1912. *Konforme Abbildung des Erdellipsoids in der Ebene. Drucj und Verlag von B.G*. Teubner, Leipzig.

Lee, L. P., 1976. *Conformal Projections Based on Elliptic Functions* (No.16). BV Gutsell.

Lauf, G. B., 1983. *Geodesy and Map Projections*. Tafe Publications, Collingwood.

Snyder, J. P., 1987. *Map Projections—A Working Manual*. US Geol Surv Prof Paper 1395, Washington D C.

Yang, Q. H., Snyder, J. P., Tobler, W. R., 2000. *Map Projection Transformation: Principles and Applications*. Taylor & Francis, London.

Revised April 2016