The Research on Renewal Theory and Method for the CGCS2000 Reference Framework

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Abstract

The CGCS2000 (China Geodetic Coordinate System 2000) reference framework, which has been employed since July 1, 2008 is based on the ITRF97 reference framework and only meets the application requirements of China's regional. With the sustained development of China's economy and society, and the globalization of the applications of BeiDou navigation satellite system (BDS), there is a need to establish global CGCS2000 reference framework. This paper studies mathematical method for construction Global CGCS2000 reference framework, the theory and algorithm of two-step method with the inner constraints theory is analyzed. The constraint conditions of coordinate reference are redefined according to the minimum standard of frame transition parameters and rate variation. As a result, the adjusted network enjoys the highest degree of fitting to the shape of the initial network and maintain the inherent purity of the coordinate network using different observation technologies, this research result can improve the basic theory of terrestrial reference framework determination, and provide scientific methods for the globalization of the CGCS2000.

1. Introduction

The CGCS2000 (China Geodetic Coordinate System 2000) reference framework, which has been employed on July 1, 2008, is based on the ITRF97 reference framework and only meets the application requirements of China's regional coordinate reference framework. With the global expansion of the BeiDou Navigation Satellite System, there is an increasing demand to establish an independent global coordinate reference framework for China. Although domestic scholars have extensively studied the establishment of international and regional reference frameworks, systematic research on the underlying theory of reference framework construction has not been conducted, nor have the implementation methods for a global CGCS2000 coordinate reference framework been thoroughly investigated. This paper research on renewal theory and method for the CGCS2000 Reference Framework, the research result can improve the basic theory of terrestrial reference framework determination, and provide scientific methods for the globalization of the CGCS2000.

2. Establishment of the Mathematical Model for ITRF

The mathematical model for constructing the ITRF（International Terrestrial Reference Frame） should possess a sound theoretical foundation while also being numerically feasible. The ITRF construction model primarily encompasses three aspects (Dermanis, 2014):

1.Coordinate Time Series Model: A mathematical model for determining the parameters a_i from the extensive valid data of ITRF stations, expressed as $x_i(t)=F(a_i,t)$. This model represents the coordinates x_i of any ITRF station i as a function of time t, through station-specific parameters a_i.

2.Coordinate Frame Construction Model: This model realizes the combination processing of station coordinates $x_{T,i}(k)$ obtained from various observation techniques T at different observation epochs k. It accomplishes the inter-technique combination and performs the transformation of ITRF parameters between the instantaneous epoch frame S_K, the different observational technique frames S_T, and the final ITRF frame S_{ITRF}.

3.Inner Constraints Algorithm: This involves selecting constraint conditions to eliminate the influence of rank deficiency in the frame transformation equations, thereby establishing an "optimal" ITRF.

The construction of a Terrestrial Reference Frame (TRF) typically employs two mathematical models. The one-step method model synchronously combines the valid time series from different space techniques (synchronous combination). The two-step method model involves, in its first step, the separate combination processing of the valid time series from different observation techniques (intra-technique combination), yielding the initial TRF parameters (initial station coordinates and linear velocities) for each technique. The second step combines the effective parameter estimates obtained from the first step for all observation techniques, achieving the joint processing of results between techniques (inter-technique combination) to produce the final TRF results.

Due to space limitations, this paper focuses solely on the two-step method model. The advantages of the two-step method over the one-step method analysed in the final section.

3. Two-Step Method for Frame Construction

3.1 Intra-Technique Combination Model

The intra-technique combination model processes the time series of different observation techniques separately, it estimates the initial coordinate and velocity values of the reference framework stations from the coordinate time series respectively. The processing model is independent of any specific observation technique and can be expressed in a general form (Golub et al., 1979).

$$\tilde{x}_i^k = x_i(t_k) = (1+s_k)\mathrm{R}(\theta_k)[\mathrm{x}_{i0} + (t_k - t_0)\mathrm{v}_i] + d_k + e_i^k \qquad (1)$$

Where s_k, θ_k, d_k are the transformation parameters from the reference frame S_T of observation technique T to the original observation frame S_k at epoch t_k. After linear approximation, it can be expressed as the following formula:

$$\tilde{x}_i^k = \mathrm{x}_{i0} + (t_k - t_0)\mathrm{v}_i + s_k \mathrm{x}_{i0} + [\mathrm{x}_{i0}\times]\theta_k + d_k + e_i^k \quad (2)$$

Using the approximate values x_{0i}^{ap}, v_i^{ap} ($s_k^{ap}=0, \theta_k^{ap}=0, d_k^{ap}=0$) and $\delta x_i^k = x_i^k - (x_i^k)^{ap} = x_i^k - [x_{i0}^{ap} + (t_k - t_0)v_i^{ap}]$, the corrections δx_{i0}, δv_i. The observation equation can be written as:

$$\begin{aligned}\delta\tilde{x}_i^k &= \delta\mathrm{x}_{i0} + (t_k - t_0)\delta\mathrm{v}_i + s_k \mathrm{x}_{i0}^{ap} + [\mathrm{x}_{i0}^{ap}\times]\theta_k + d_k + e_i^k \\ &= \begin{bmatrix} I & (t_k - t_0)\end{bmatrix}\begin{bmatrix}\delta\mathrm{x}_{i0} \\ \delta\mathrm{v}_i\end{bmatrix} + \begin{bmatrix}[\mathrm{x}_{i0}^{ap}\times] & I & \mathrm{x}_{i0}^{ap}\end{bmatrix}\begin{bmatrix}\theta_k \\ d_k \\ s_k\end{bmatrix} + e_i^k\end{aligned} \quad (3)$$

The observation equation above can be transformed into the compact form:

$$\delta x_i^k = A_{ai}a_i + A_{zi}z_k + e_i^k \quad (4)$$

Here, $a_i = \begin{bmatrix}\delta\mathrm{x}_{i0} \\ \delta\mathrm{v}_i\end{bmatrix}$ represents the ITRF parameters for station i, namely δx_{i0}, δv_i. Z_k contains the transformation parameters from the reference frame S_T of observation technique T to the coordinate frame S_k at the independent observation epoch t_k, where $z_k = \begin{bmatrix}\theta_k \\ d_k \\ s_k\end{bmatrix}$, $A_{ai} = \begin{bmatrix} I & (t_k - t_0)\end{bmatrix}$, $A_{zi} = \begin{bmatrix}[\mathrm{x}_{i0}^{ap}\times] & I & \mathrm{x}_{i0}^{ap}\end{bmatrix} = E_i$.

The parameters to be estimated comprise the corrections to the approximate values of the initial coordinates and station velocities, δx_{i0}, $\delta \mathrm{v}_i$ as well as the transformation parameters from the ITRF to the reference frame at the observation epoch, namely the rotation angle θ_k, translation parameter d_k, and scale parameter s_k.

The estimation of frame transformation parameters should account for the differences between epochs t_k, which forms the basis of the combination processing procedure. In practice, within each space technique, processing results differ across various epochs t_k (e.g., day, week, session) within a time cycle. During the time cycle at t_k, the unknown parameters x_i, Earth Orientation Parameters (EOPs), etc., within each technique are assumed to be constant.

The covariance matrix of the input coordinates x_k at each epoch t_k is given by $\hat{c}_k = \hat{c}_{x_k} = \hat{\sigma}_k^2 Q_{x_k}$, where $\hat{\sigma}_k^2$ is the variance factor, and the cofactor matrix Q_{x_k} is a submatrix of the cofactor matrix Q_k for all parameters. If the correct Q_{x_k} is obtained, it should be a singular matrix, and the weight matrix $P_{x_k} = Q_{x_k}^{-1}$ cannot be obtained through direct inversion. Therefore, prior coordinate information (prior covariance matrix $C_{x_k}^{prior} = \sigma_{prior}^2 I$) is typically utilized to obtain an invertible Q_{x_k} , and subsequently $P_{x_k} = Q_{x_k}^{-1}$. Since observations at two different epochs are uncorrelated, the stacked weight matrix is a block-diagonal matrix of P_{x_k} . By assembling observations from all stations in each epoch, the observation equation for N_k stations at epoch k is obtained as follows (same epoch, different stations).

$$\begin{aligned}\delta x_k &= \begin{bmatrix}\delta x_1^k \\ \vdots \\ \delta x_i^k \\ \vdots \\ \delta x_{N_k}^k\end{bmatrix} = \begin{bmatrix}x_{0,1} \\ \vdots \\ x_{0,i} \\ \vdots \\ x_{0,N_k}\end{bmatrix} + (t_k - t_0)\begin{bmatrix}v_1 \\ \vdots \\ v_i \\ \vdots \\ v_{N_k}\end{bmatrix} + \begin{bmatrix}E_1 \\ \vdots \\ E_i \\ \vdots \\ E_{N_k}\end{bmatrix} Z_k + \begin{bmatrix}e_1^k \\ \vdots \\ e_i^k \\ \vdots \\ e_{N_k}^k\end{bmatrix} \\ &= x_{0(k)} + (t_k - t_0)v_{(k)} + E_{(k)}Z_k + e_k\end{aligned} \quad (5)$$

where $x_{0(k)}$ and $v_{(k)}$ represent the collection of initial coordinates x_0 and velocities v for all N stations at epoch t_k.

Cases where stations lack data at individual epochs can be handled using the accompanying matrix L_k. The $N_k \times N$ matrix L_k is an identity matrix corresponding to the valid stations at epoch t_k. Thus, the observation equation at epoch t_k is expressed as the following formula:

$$\delta x_k = L_k x_0 + (t_k - t_0)L_k v_{(k)} + L_k E Z_k + e_k \quad (6)$$

Finally, the combination processing equation for the first step is obtained by combining M epochs, each containing N stations.

$$\begin{aligned} b = \delta x &= \begin{bmatrix}\delta x_1 \\ \vdots \\ \delta x_k \\ \vdots \\ \delta x_M\end{bmatrix} \\ &= \begin{bmatrix}L_1 \\ \vdots \\ L_k \\ \vdots \\ L_M\end{bmatrix} x_0 + \begin{bmatrix}(t_1 - t_0)L_1 \\ \vdots \\ (t_k - t_0)L_k \\ \vdots \\ (t_M - t_0)L_M\end{bmatrix} v + \begin{bmatrix}L_1E & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & L_kE & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & L_ME\end{bmatrix}\begin{bmatrix}z_1 \\ \vdots \\ z_k \\ \vdots \\ z_M\end{bmatrix} + \begin{bmatrix}e_1 \\ \vdots \\ e_k \\ \vdots \\ e_M\end{bmatrix} \\ &= A_{x_0}x_0 + A_v v + A_z z + e = [A_{x_0}\ A_v\ A_z]\begin{bmatrix}x_0 \\ v \\ z\end{bmatrix} + e = Ax + e \\ &= [A_{x_0}\ A_v]\begin{bmatrix}x_0 \\ v\end{bmatrix} + A_z z + e = A_a a + A_z z + e = [A_a\ A_z]\begin{bmatrix}a \\ z\end{bmatrix} + e = Ax + e\end{aligned} \quad (7)$$

$$A_a = \begin{bmatrix}A_{x_0} & A_v\end{bmatrix}, \quad a = \begin{bmatrix}x_0 \\ v\end{bmatrix} \quad (8)$$

The weight matrix is expressed as follows:

$$P = \begin{bmatrix}P_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & P_k & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & P_M\end{bmatrix} \quad (9)$$

The normal equation is $N\hat{x} = u$, where $N=A^TPA$ and $u=A^TPb$. Since the normal equation matrix N is singular, it does not have a unique solution. N and u are expressed as follows:

$$\begin{aligned} N &= \begin{bmatrix}N_a & N_{az} \\ N_{za} & N_z\end{bmatrix} = \begin{bmatrix}N_{x_0} & N_{x_0v} & N_{x_0z} \\ N_{vx_0} & N_v & N_{vz} \\ N_{zx_0} & N_{zv} & N_z\end{bmatrix} \\ &= \begin{bmatrix}\sum_{k=1}^{M}\bar{P}_k & \sum_{k=1}^{M}(t_k - t_0)\bar{P}_k & (\sum_{k=1}^{M}\bar{P}_k)E \\ & \sum_{k=1}^{M}(t_k - t_0)^2\bar{P}_k & (\sum_{k=1}^{M}(t_k - t_0)\bar{P}_k)E \\ symm & & E^{\mathrm{T}}(\sum_{k=1}^{M}\bar{P}_k)E\end{bmatrix}\end{aligned} \quad (10)$$

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$$u=\begin{bmatrix}u_a\\u_z\end{bmatrix}=\begin{bmatrix}u_{x_0}\\u_v\\u_z\end{bmatrix}=\begin{bmatrix}\sum_{k=1}^{M}u_k\\\sum_{k=1}^{M}(t_k-t_0)u_k\\E^T\sum_{k=1}^{M}u_k\end{bmatrix},u_k=L_k^TP_k\triangle x_k, \quad (11)$$

Here, $\overline{P}_k=L_k^TP_kL_k$ is a weight matrix of size $N\times N$. When a station lacks observations at epoch t_k, zero elements must be added to the k-th row and k-th column of P_k. The matrix N in the normal equation $N\hat{x}=u$ is singular. The rank deficiency arises because the transformation parameters of the coordinate frame are undefined. The infinite number of solutions to the normal equation corresponds to the infinite choices of the reference frame. Additional minimum constraints must be applied to obtain unique parameter estimates and thus define a specific reference frame.

3.2 Solution of the Combined Normal Equation

The solution of the normal equation requires the joint estimation of transformation parameters. Based on Equation(7), Equation (12)is obtained.

$$\begin{bmatrix}N_a & N_{az}\\N_{za} & N_z\end{bmatrix}\begin{bmatrix}\hat{a}_{T|T}\\\hat{z}_{T|T}\end{bmatrix}=\begin{bmatrix}u_a\\u_z\end{bmatrix} \quad (12)$$

It should be noted that the subscripts $T|T$, such as $V|V$, $S|S$, $G|G$, $D|D$, $C|C$ represent VLBI, SLR, GPS, DORIS, and COMPASS/BeiDou, respectively. The separator | indicates that the parameter estimation occurs only within one specific technique $T|$, based on the observations from that particular technique $|T$.

The normal equation can be decomposed into two parts: $N_a\hat{a}_{T|T}+N_{az}\hat{z}_{T|T}=u_a$, $N_{za}\hat{a}_{T|T}+N_z\hat{z}_{T|T}=u_z$. First, by evaluating the redundant parameters $\hat{z}_{T|T}$, we can extract $\hat{z}_{T|T}$ from the second equation and substitute it into the first equation to obtain the single-difference normal equation.

$$\overline{N}_a\hat{a}_{T|T}=\overline{u}_a \quad (13)$$

where $\overline{N}_a=N_a\text{-}N_{az}N_z^{-1}N_{za}$, $\overline{u}_a=u_a\text{-}N_{az}N_z^{-1}u_z$

$$\begin{aligned}&N_a\hat{a}_{T|T}+N_{az}\hat{z}_{T|T}=u_a,\ \ N_{za}\hat{a}_{T|T}+N_z\hat{z}_{T|T}=u_z\\&N_z\hat{z}_{T|T}=u_z\text{-}N_{za}\hat{a}_{T|T},\ \ \hat{z}_{T|T}=u_zN_z^{-1}\text{-}N_{za}\hat{a}_{T|T}N_z^{-1}\\&N_a\hat{a}_{T|T}+N_{az}(u_zN_z^{-1}\text{-}N_{za}\hat{a}_{T|T}N_z^{-1})=u_a\\&N_a\hat{a}_{T|T}+N_{az}u_zN_z^{-1}\text{-}N_{az}N_{za}\hat{a}_{T|T}N_z^{-1}=u_a\end{aligned} \quad (14)$$

This simplified normal equation possesses the same rank deficiency as the original equation. A unique solution for the parameters can only be obtained through a least squares solution involving only the parameters a. Once the unique solution is obtained, the corresponding parameters z can be estimated using the following formula:

$$\hat{z}_{T|T}=N_z^{-1}u_z\text{-}N_z^{-1}N_{za}\hat{a}_{T|T} \quad (15)$$

The inter-technique combination processing model utilizes the initial station coordinates and velocity values $(\delta\hat{x}_{i0})_{T|T},(\delta\hat{v}_i)_{T|T},i=1_T,2_T,\cdots,n_T$, for each observation technique T, and the transformation parameters $(\hat{z}_k)_{T|T},k_T=1_T,2_T,\cdots m_T$ from the observational technique frame S_T to each observation epoch $t_{1_T},\cdots,t_{k_T},\cdots,t_{m_T}$ frame S_k, which were obtained during the intra-technique combination stage. These serve as observations in the combination processing stage for the joint estimation of the global CGCS2000 frame model parameters $\delta\hat{x}_{i0},\delta\hat{v}_i$ and the transformation parameters.

$$\begin{aligned}(\delta\tilde{x}_{0i})_{T|T}&=\delta\mathbf{x}_{0i}+[\mathbf{x}_{0i}^{ap}\times]\theta_{0T}+S_{0T}\mathbf{x}_{0i}^{ap}+d_{0T}+e_{(\delta\hat{x}_{0i})_{T|T}}\\&=\delta\mathbf{x}_{0i}+\begin{bmatrix}[\mathbf{x}_{0i}^{ap}\times] & I & \mathbf{x}_{0i}^{ap}\end{bmatrix}\begin{bmatrix}\theta_{0T}\\d_{0T}\\S_{0T}\end{bmatrix}+e_{(\delta\hat{x}_{0i})_{T|T}}\\&=\delta\mathbf{x}_{0i}+E_ip_{0T}+e_{(\delta\hat{x}_{0i})_{T|T}},\ \ i=1_T,2_T,n_T\end{aligned} \quad (16)$$

$$\begin{aligned}(\delta\tilde{v}_i)_{T|T}&=\delta v_i+[\mathbf{x}_{0i}^{ap}\times]\dot{\theta}_T+\dot{S}_T\mathbf{x}_{0i}^{ap}+\dot{d}_T+e_{(\delta\hat{v}_i)_{T|T}}\\&=\delta v_i+\begin{bmatrix}[\mathbf{x}_{0i}^{ap}\times] & I & \mathbf{x}_{0i}^{ap}\end{bmatrix}\begin{bmatrix}\dot{\theta}_T\\\dot{d}_T\\\dot{S}_T\end{bmatrix}+e_{(\delta\hat{v}_i)_{T|T}}\\&=\delta v_i+E_i\dot{p}_T+e_{(\delta\hat{v}_i)_{T|T}},\ \ i=1_T,2_T,n_T\end{aligned} \quad (17)$$

$(\hat{z}_k)_{T|T}$ Represents the transformation parameters from reference frame B of observation technique T to reference frame C at observation epoch t_k (where the transformation from frame B to frame C is known). To estimate the transformation parameters $z_{k,T}$ from the ITRF reference frame (A) to reference frame C at observation epoch t_k, the relationship among the transformation parameters of the three frames is utilized: $P_{A\rightarrow C}=P_{A\rightarrow B}+P_{B\rightarrow C}$, where $P_{A\rightarrow C}=z_{k,T}$, $P_{A\rightarrow B}=P_T(t_k)$, $P_{B\rightarrow C}=(\hat{z}_k)_{T|T}$, $z_{k,T}=(\hat{z}_k)_{T|T}+P_T(t_k)$ and $P_T(t_k)=P_{0T}+(t_k\text{-}t_0)\dot{P}_T$. The resulting observation equation is:

$$(\hat{z}_k)_{T|T}=z_{k,T}\text{-}P_{0T}\text{-}(t_k\text{-}t_0)\dot{P}_T+e_{(\hat{z}_k)_{T|T}} \quad (18)$$

Each observation technique inherently possesses a specific spatial datum and temporal datum. For observations from a single technique, the datum can be considered unified. However, the datums across multiple observation techniques are often inconsistent. This datum inconsistency inevitably affects the results of observational data fusion. Therefore, the issue of datum unification must be addressed in the integrated processing of different types of observables. Different space observation techniques form different observation networks. The subscripts V, S, G, D, and C represent VLBI, SLR, GPS, DORIS, and COMPASS/BeiDou, respectively; i represents the station, and k represents the epoch. The sequence numbers 1_T, 2_T, i_T, ..., n_T represent the transformation from the specified epoch frame sequence to the ITRF.

Based on Equations (16), (17), and (18), the observation equations are established using all fundamental stations from all observation techniques. This allows for the final estimation of the unknown parameters δx_{0i}, δv_i, $Z_{k,T}$ (A to C), and the "redundant" parameters p_{0T}, $\dot{p}_T$ (transformation parameters from the global CGCS2000 reference frame (A) to the observation technique frame (B) and their rates $\theta_{0T},\dot{\theta}_T,d_{0T},\dot{d}_T,S_{0T},\dot{S}_T$).

The observations consist of the initial coordinates $(\delta\hat{x}_{0i})_{T|T}$, , velocity values $(\delta\hat{v}_i)_{T|T}$, and transformation parameters $(\hat{z}_k)_{T|T}$ (B to C) for station i. During the combination processing stage of different observation techniques, independent inner constraints are selected to define the distinct coordinate frame parameters for each technique, namely $(\delta\hat{x}_{0i})_{T|T}$, $(\delta\hat{v}_i)_{T|T}$, $(\hat{z}_k)_{T|T}$.

The unknown parameters are the corrections δx_{0i}, δv_i for the stations in the integrated frame network, which represent the joint solution derived from the observation network of each technique and the transformation parameters $Z_{k,T}$. Here, $Z_{k,T}$ denotes the transformation parameters from the ITRF reference frame to all observation epochs t_k within each observation technique T. All unknown parameters collectively define the new common ITRF reference frame. Furthermore, the parameters $Z_{k,T}$ are not "redundant" parameters; they enable the transformation of Earth Orientation Parameters (EOPs) from the initial ITRF frame to the observation epoch t_k frames of all techniques T.

The combined observation equation for all stations in observation technique T is as follows:

$$(\delta\hat{x}_{0i})_{T|T}=\begin{bmatrix}(\delta\hat{x}_{01})_{T|T}\\ \vdots\\ (\delta\hat{x}_{0i})_{T|T}\\ \vdots\\ (\delta\hat{x}_{0n})_{T|T}\end{bmatrix}=\begin{bmatrix}\delta x_{01}\\ \vdots\\ \delta x_{0i}\\ \vdots\\ \delta x_{0n_T}\end{bmatrix}+\begin{bmatrix}E_1\\ \vdots\\ E_i\\ \vdots\\ E_{n_T}\end{bmatrix}P_{T0}+\begin{bmatrix}e_{(\delta\hat{x}_{01})_{T|T}}\\ \vdots\\ e_{(\delta\hat{x}_{0i})_{T|T}}\\ \vdots\\ e_{(\delta\hat{x}_{0n})_{T|T}}\end{bmatrix} \quad (19)$$
$$=\delta\hat{x}_{0T}+E_{xT}P_{0T}+e_{(\delta\hat{x}_0)_{T|T}}$$

$$(\delta\hat{v})_{T|T}=\begin{bmatrix}(\delta v_1)_{T|T}\\ \vdots\\ (\delta v_i)_{T|T}\\ \vdots\\ (\delta v_n)_{T|T}\end{bmatrix}=\begin{bmatrix}\delta v_1\\ \vdots\\ \delta v_i\\ \vdots\\ \delta v_{n_T}\end{bmatrix}+\begin{bmatrix}E_1\\ \vdots\\ E_i\\ \vdots\\ E_{n_T}\end{bmatrix}\dot{P}_T+\begin{bmatrix}e_{(\delta\hat{v}_1)_{T|T}}\\ \vdots\\ e_{(\delta\hat{v}_i)_{T|T}}\\ \vdots\\ e_{(\delta\hat{v}_n)_{T|T}}\end{bmatrix} \quad (20)$$
$$=\delta v_T+E_{xT}\dot{P}_T+e_{(\delta\hat{v})_{T|T}}$$

After combining Equations (19) and (20), it is expressed as:

$$\hat{a}_{T|T}=\begin{bmatrix}(\delta\hat{x}_0)_{T|T}\\ \delta\hat{v}_{T|T}\end{bmatrix}=\begin{bmatrix}\delta x_{0T}\\ \delta v_T\end{bmatrix}+\begin{bmatrix}E_{xT} & 0\\ 0 & E_{xT}\end{bmatrix}\begin{bmatrix}P_{0T}\\ \dot{P}_T\end{bmatrix}+\begin{bmatrix}e_{(\delta\hat{x}_0)_{T|T}}\\ e_{\delta\hat{v}_{T|T}}\end{bmatrix} \quad (21)$$
$$=a_T+E_{aT}P_T+e_{\hat{a}_{T|T}}$$

To facilitate the implementation of the Inner Constraints Algorithm, a separate observation equation encompassing all epochs is constructed specifically for the parameters $\hat{z}_{T|T}$.

$$\hat{z}_{T|T}=\begin{bmatrix}(\hat{z}_1)_{T|T}\\ \vdots\\ (\hat{z}_k)_{T|T}\\ \vdots\\ (\hat{z}_m)_{T|T}\end{bmatrix}=\begin{bmatrix}z_{1,T}\\ \vdots\\ z_{k,t}\\ \vdots\\ z_{m,t}\end{bmatrix}-\begin{bmatrix}I & (t_1-t_0)I\\ \vdots & \vdots\\ I & (t_k-t_0)I\\ \vdots & \vdots\\ I & (t_m-t_0)I\end{bmatrix}\begin{bmatrix}p_{0s}\\ \dot{p}_s\end{bmatrix}+\begin{bmatrix}e_{(\hat{z}_1)_{T|T}}\\ \vdots\\ e_{(\hat{z}_k)_{T|T}}\\ \vdots\\ e_{(\hat{z}_m)_{T|T}}\end{bmatrix} \quad (22)$$
$$=z_T+E_{zT}P_T+e_{\hat{z}_{T|T}}$$

The inter-technique combination processing observation equation, which includes all observation techniques, is as follows:

$$\hat{a}_{\mathrm{P|P}}=\begin{bmatrix}\hat{a}_{V|V}\\ \hat{a}_{S|S}\\ \hat{a}_{G|G}\\ \hat{a}_{D|D}\end{bmatrix}=\begin{bmatrix}a_V\\ a_S\\ a_G\\ a_D\end{bmatrix}+\begin{bmatrix}E_{aV} & 0 & 0 & 0\\ 0 & E_{aS} & 0 & 0\\ 0 & 0 & E_{aG} & 0\\ 0 & 0 & 0 & E_{aD}\end{bmatrix}\begin{bmatrix}P_V\\ P_S\\ P_G\\ P_D\end{bmatrix}+\begin{bmatrix}e_{\hat{a}_{V|V}}\\ e_{\hat{a}_{S|S}}\\ e_{\hat{a}_{G|G}}\\ e_{\hat{a}_{D|D}}\end{bmatrix}=a+E_a p+e_{\hat{a}_T} \quad (23)$$

$$\hat{z}_{\mathrm{p|p}}=\begin{bmatrix}\hat{z}_{V|V}\\ \hat{z}_{S|S}\\ \hat{z}_{G|G}\\ \hat{z}_{D|D}\end{bmatrix}=\begin{bmatrix}z_V\\ z_S\\ z_G\\ z_D\end{bmatrix}+\begin{bmatrix}E_{zV} & 0 & 0 & 0\\ 0 & E_{zS} & 0 & 0\\ 0 & 0 & E_{zG} & 0\\ 0 & 0 & 0 & E_{zD}\end{bmatrix}\begin{bmatrix}P_V\\ P_S\\ P_G\\ P_D\end{bmatrix}+\begin{bmatrix}e_{\hat{z}_{V|V}}\\ e_{\hat{z}_{S|S}}\\ e_{\hat{z}_{G|G}}\\ e_{\hat{z}_{D|D}}\end{bmatrix}=a+E_z p+e_{\hat{z}_T} \quad (24)$$

The inter-technique combination processing observation equation is expressed in compact form as:

$$b_T=\begin{bmatrix}\hat{a}_{\mathrm{p|p}}\\ \hat{z}_{\mathrm{p|p}}\end{bmatrix}=\begin{bmatrix}I & 0 & E_a\\ 0 & I & E_z\end{bmatrix}\begin{bmatrix}a\\ z\\ p\end{bmatrix}+\begin{bmatrix}e_{\hat{a}_{\mathrm{p|p}}}\\ e_{\hat{z}_{\mathrm{p|p}}}\end{bmatrix}=A_T X+e_T \quad (25)$$

The corresponding form of the weight matrix is:

$$P_T=\begin{bmatrix}P_{a_T} & P_{a_T Z_T}\\ P^T_{a_T Z_T} & P_{Z_T}\end{bmatrix} \quad (26)$$

where

$$P_a=\begin{bmatrix}P_{a_V} & 0 & 0 & 0\\ 0 & P_{a_S} & 0 & 0\\ 0 & 0 & P_{a_G} & 0\\ 0 & 0 & 0 & P_{a_D}\end{bmatrix}$$
$$P_z=\begin{bmatrix}P_{z_V} & 0 & 0 & 0\\ 0 & P_{z_S} & 0 & 0\\ 0 & 0 & P_{z_G} & 0\\ 0 & 0 & 0 & P_{z_D}\end{bmatrix} \quad (27)$$
$$P_{az}=\begin{bmatrix}P_{a_V z_V} & 0 & 0 & 0\\ 0 & P_{a_S z_S} & 0 & 0\\ 0 & 0 & P_{a_G z_G} & 0\\ 0 & 0 & 0 & P_{a_D z_D}\end{bmatrix}$$

During the inter-technique combination stage, the results from various space observation techniques must be integrated through a combined adjustment to obtain the global CGCS2000 frame parameters for the observing stations. However, the observation networks of different space techniques lack common observation points, and the observation equations do not contain common unknowns. The current common approach utilizes co-location sites, which involves selecting stations from two or more different observation techniques that are in close proximity. The vector values between stations of different techniques T_1, T_2, obtained through high-precision measurements, are referred to as local ties (Z Altamimi, 2007).

$$d_{T_1,T_2}(t)=x_{T_2}(\mathrm{t})-x_{T_1}(\mathrm{t})=[x_{0T_2}-x_{0T_1}]+(t-t_0)[v_{T_2}-v_{T_1}] \quad (28)$$

By performing repeated observations at different time epochs t, the initial value and rate of change of the local ties can be estimated, $d^0_{T_1,T_2}=x_{0T_2}-x_{0T_1}$, $\dot{d}_{T_1,T_2}=v_{T_2}-v_{T_1}$.

The observation equation for local ties is:

$$b_C=A_{CV}a_V+A_{CS}a_S+A_{CG}a_G+A_{CD}a_D+e_C$$
$$=\begin{bmatrix}A_{CV} & A_{CS} & A_{CG} & A_{CD}\end{bmatrix}\begin{bmatrix}a_V\\ a_S\\ a_G\\ a_D\end{bmatrix}+e_C=A_C a+e_C \quad (29)$$

Normal equations are independently established for each observation technique $\hat{a}_{V|V},\hat{a}_{S|S},\hat{a}_{G|G},\hat{a}_{D|D}$: $N_T x=u_T$. The normal equation for co-location sites b_C is $N_C x=u_C$. The observation equations should include the transformation parameters z, p.

$$b_C=A_C a+e_C=\begin{bmatrix}A_C & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ z\\ p\end{bmatrix}=\begin{bmatrix}A_C & 0 & 0\end{bmatrix}x+e_C \quad (30)$$

From Formula (25), the normal equation composed of different space observations is obtained as $N_T\hat{X}=(\mathrm{A}_T^T\mathrm{P}_T\mathrm{A}_T)\hat{X}=\mathrm{A}_T^T\mathrm{P}_T b_T=u_T$. Here, N_T and u_T are expressed as:

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$$\begin{aligned}N_T &= \mathrm{A}_T^T \mathrm{P}_T \mathrm{A}_T = \begin{bmatrix} I & 0 \\ 0 & I \\ E_a^T & E_Z^T \end{bmatrix} \begin{bmatrix} P_a & P_{az} \\ P_{az}^T & P_z \end{bmatrix} \begin{bmatrix} I & 0 & E_a \\ 0 & I & E_z \end{bmatrix} \\ &= \begin{bmatrix} P_a & P_{\mathrm{az}} & P_a E_a + P_{az} E_z \\ P_{az}^T & P_z & P_{az}^T E_a + P_z E_z \\ E_a^T P_a + E_z^T P_{az}^T & E_a^T P_{az} + E_z^T P_z & E_a^T P_a E_a + E_z^T P_{az}^T E_a + E_a^T P_{az} E_z + E_z^T P_z E_z \end{bmatrix} \\ &= \begin{bmatrix} N_a & N_{az} & N_{ap} \\ N_{az}^T & N_z & N_{zp} \\ N_{ap}^T & N_{zp}^T & N_p \end{bmatrix}\end{aligned} \tag{31}$$

$$\begin{aligned}u_T &= \mathrm{A}_T^T \mathrm{P}_T\, b_T = \begin{bmatrix} I & 0 \\ 0 & I \\ E_a^T & E_Z^T \end{bmatrix} \begin{bmatrix} P_a & P_{az} \\ P_{az}^T & P_z \end{bmatrix} \begin{bmatrix} \hat{a}_T \\ \hat{z}_T \end{bmatrix} \\ &= \begin{bmatrix} P_a \hat{a}_T + P_{az} \hat{z}_T \\ P_{az}^T \hat{a}_T + P_z \hat{z}_T \\ E_a^T P_a \hat{a}_T + E_z^T P_{az}^T \hat{a}_T + E_a^T P_{az} \hat{z}_T + E_z^T P_z \hat{z}_T \end{bmatrix} = \begin{bmatrix} u_a \\ u_z \\ u_p \end{bmatrix}\end{aligned} \tag{32}$$

The normal equation for co-location sites is:

$$\begin{aligned}\left(\begin{bmatrix} A_C^T \\ 0 \\ 0 \end{bmatrix} P_C \begin{bmatrix} A_C & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{a} \\ \hat{z} \\ \hat{p} \end{bmatrix} &= \begin{bmatrix} A_C^T P_C A_C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{z} \\ \hat{p} \end{bmatrix} \\ &= \begin{bmatrix} A_C^T \\ 0 \\ 0 \end{bmatrix} P_c b_c = \begin{bmatrix} A_c^T P_c b_c \\ 0 \\ 0 \end{bmatrix}\end{aligned} \tag{33}$$

The two-step method combined processing normal equation we derived is:

$$\begin{bmatrix} N_a + N_c & N_{az} & N_{ap} \\ N_{az}^T & N_z & N_{zp} \\ N_{ap}^T & N_{zp}^T & N_p \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{z} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} u_a + u_c \\ u_z \\ u_p \end{bmatrix} \tag{34}$$

Where $N_c = A_C^T P_c A_c, u_c = A_C^T P_c b_c$.

3.3 Solution of the Combined Normal Equation

The solution of the normal equation can begin by eliminating the "redundant" parameters z or p. For instance, when first eliminating parameter z, the process shares similarities with the combination stage of the two-step method. The normal equation in (34) is decomposed into three subsets:

$$\begin{aligned}&(N_a + N_c)\hat{a} + N_{az}\hat{z} + N_{ap}\hat{p} = u_a + u_c, \\ &N_{az}^T \hat{a} + N_z \hat{z} + N_{zp}\hat{p} = u_z, \\ &N_{ap}^T \hat{a} + N_{zp}^T \hat{z} + N_p \hat{p} = u_p\end{aligned} \tag{35}$$

Solve for $\hat{z}$ from the second equation and substitute it into the other two equations, obtaining the single-difference normal equation:

$$\begin{bmatrix} \bar{N}_a + N_c & \bar{N}_{ap} \\ \bar{N}_{ap}^T & \bar{N}_p \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} \bar{u}_a + u_c \\ \bar{u}_p \end{bmatrix} \tag{36}$$

Where $\bar{N}_a = N_a - N_{az} N_z^{-1} N_{az}^T$, $\bar{N}_{ap} = N_{ap} - N_{az} N_z^{-1} N_{zp}$, $\bar{N}_p = N_p - N_{zp}^T N_z^{-1} N_{zp}$, $\bar{u}_a = u_a - N_{az} N_z^{-1} u_z$, $\bar{u}_p = u_p - N_{zp}^T N_z^{-1} u_z$.

The single-difference normal equation possesses the same rank deficiency as the original normal equation, corresponding to an infinite number of solutions for different coordinate reference frames. Minimum constraints must be applied to obtain a unique solution. Since the single-difference normal equation contains only parameters a and p, once the unique estimates for parameters $\hat{a}$ and $\hat{p}$ are obtained, parameter $\hat{z}$ can be estimated using the following formula:

$$\hat{z} = N_z^{-1} u_z - N_z^{-1} N_{az}^T \hat{a} - N_z^{-1} N_{zp} \hat{p} \tag{37}$$

The parameter $\hat{p}$ can be further eliminated by using the double-difference normal equation. Decomposing equation (36) yields:

$$(\bar{N}_a + N_c)\hat{a} + \bar{N}_{ap} = \bar{u}_a + u_c, \bar{N}_{ap}^T \hat{a} + \bar{N}_p \hat{p} = \bar{u}_p \tag{38}$$

Parameter $\hat{p}$ can be solved from the second equation and substituted into the first equation, resulting in the following double-difference normal equation:

$$\begin{aligned}(\bar{N}_a - \bar{N}_{ap} \bar{N}_p^{-1} \bar{N}_{ap}^T + N_c)\hat{a} &= (\bar{\bar{N}}_a + N_c)\hat{a} \\ &= \bar{u}_a - \bar{N}_{ap} \bar{N}_p^{-1} \bar{u}_p + u_c = \bar{\bar{u}}_a + u_c\end{aligned} \tag{39}$$

In the above equation, $\bar{\bar{N}}_a = \bar{N}_a - \bar{N}_{ap} \bar{N}_p^{-1} \bar{N}_{ap}^T$, $\bar{\bar{u}}_a = \bar{u}_a - \bar{N}_{ap} \bar{N}_p^{-1} \bar{u}_p$. By applying minimum constraints involving only the unknown parameters a, the double-difference normal equation $(\bar{\bar{N}}_a + N_c)\hat{a} = \bar{\bar{u}}_a + u_c$ is solved to obtain the unique estimate for parameter $\hat{a}$. Using equation (39), the estimates for the other parameters are then sequentially obtained:

$$\hat{z} = N_z^{-1} u_z - N_z^{-1} N_{az}^T \hat{a} - N_z^{-1} N_{zp} \hat{p} \tag{40}$$

4. Inner Constraints Algorithm

The coordinate frame transformation equation, formed by the two-step mathematical models, possesses a coefficient matrix whose column rank deficiency equals the number of transformation parameters to be solved. The set of all least squares solutions for the coordinate frame network shares the same "network shape" but lacks uniqueness (Blewitt and Lavallée, 2002; Sillard and Boucher, 2001; Williams and Willis, 2006). The traditional approach to this problem involves imposing a set of external constraints (where the number of independent constraints equals the rank deficiency) to define the frame. For example, this could be the coordinates of two stations in a triangulation network, or one station coordinate and one azimuth in a traverse. For space observation techniques, prior coordinate values for at least three stations are typically required. By applying strong constraints to the known station positions and velocities, a unique network solution for the rank-deficient normal equation can be estimated. However, if the constraints are not appropriately chosen, they can distort the network shape and introduce bias into the parameter estimates. The inner constraints method is based on the minimization condition of the global CGCS2000 frame parameters and the three transformation parameters (translation, rotation, and scale) along with their rates. This approach ensures the numerical stability of the results, preserves the intrinsic purity of reference frames derived from different observation techniques, and fully exploits the strengths of each technique (e.g., the geocenter information from SLR, the scale from VLBI, etc.).

4.1 Inner Constraints Model for Intra-Technique Combination

To determine the inner constraints matrix, the functional relationship between the frame station parameters and the variations in the frame transformation—namely the rotation

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angle $\psi(t)=\psi_0+(t\text{-}t_0)\dot{\psi}$, the scale parameter $\lambda(t)=\lambda_0+(t\text{-}t_0)\dot{\lambda}$, and the displacement $g(t)=g_0+(t\text{-}t_0)\dot{g}$ —must be established based on the CGCS2000 parameter transformation model $\tilde{x}=x+Ep$ (Sillard, 2001).

The transformation equation for each station can be combined and expressed as:

$$\begin{aligned}\tilde{a}_i &= \begin{bmatrix}\delta\tilde{x}_{0i}\\ \delta\tilde{v}_i\end{bmatrix} \approx \begin{bmatrix}\delta x_{0i}+[x_{0i}^{ap}\times]\psi_0+\lambda_0 x_{0i}^{ap}+g_0\\ \delta v_i+[x_{0i}^{ap}\times]\dot{\psi}+\dot{\lambda}x_{0i}^{ap}+\dot{g}\end{bmatrix}\\ &= \begin{bmatrix}\delta x_{0i}\\ \delta v_i\end{bmatrix}+\begin{bmatrix}[x_{0i}^{ap}\times] & I & x_{0i}^{ap} & 0 & 0 & 0\\ 0 & 0 & 0 & [x_{0i}^{ap}\times] & I & x_{0i}^{ap}\end{bmatrix}\begin{bmatrix}\psi_0\\ g_0\\ \lambda_0\\ \dot{\psi}\\ \dot{g}\\ \dot{\lambda}\end{bmatrix}\\ &= a_i+E_{a_i}p\end{aligned} \tag{41}$$

To estimate the variations of the so-called redundant parameters P_k during the coordinate frame transformation, we establish the reverse frame transformation equation:

$$\begin{aligned}\delta x_{0i} &\approx \delta\tilde{x}_{0i}-[x_{0i}^{ap}\times]\psi_0-\lambda_0 x_{0i}^{ap}-g_0\\ \delta v_i &\approx \delta\tilde{v}_i-[x_{0i}^{ap}\times]\dot{\psi}-\dot{\lambda}x_{0i}^{ap}-\dot{g}\end{aligned} \tag{42}$$

Substituting the above into Equation (3) yields:

$$\begin{aligned}\delta x_i^k &= \delta x_{0i}+(t_k\text{-}t_0)\delta v_i+s_k x_{0i}^{ap}+[x_{0i}^{ap}\times]\theta_k+d_k+e_i^k\\ &= \delta\tilde{x}_{0i}-[x_{0i}^{ap}\times]\psi_0-\lambda_0 x_{0i}^{ap}-g_0+(t_k\text{-}t_0)\left\{\delta\tilde{v}_i-[x_{0i}^{ap}\times]\dot{\psi}-\dot{\lambda}x_{0i}^{ap}-\dot{g}\right\}\\ &\quad +s_k x_{0i}^{ap}+[x_{0i}^{ap}\times]\theta_k+d_k+e_i^k\\ &= \delta\tilde{x}_{0i}+(t_k\text{-}t_0)\delta\tilde{v}_i+[x_{0i}^{ap}\times]\left[\theta_k-\psi_0-(t_k\text{-}t_0)\dot{\psi}\right]\\ &\quad +\left[s_k-\lambda_0-(t_k\text{-}t_0)\dot{\lambda}\right]x_{0i}^{ap}+\left[d_k-g_0-(t_k\text{-}t_0)\dot{g}\right]+e_i^k\end{aligned} \tag{43}$$

Transforming the above equation gives:

$$\delta x_i^k=\delta\tilde{x}_{0i}+(t_k\text{-}t_0)\delta\tilde{v}_i+[x_{0i}^{ap}\times]\tilde{\theta}_k+\tilde{s}_k x_{0i}^{ap}+\tilde{d}_k+e_i^k \tag{44}$$

Comparing Equations (43) and (44), the relationship between the different transformation parameters is identified.

$$\begin{aligned}\tilde{\theta}_k &= \theta_k-\psi_0-(t_k\text{-}t_0)\dot{\psi},\\ \tilde{d}_k &= d_k-g_0-(t_k\text{-}t_0)\dot{g},\\ \tilde{s}_k &= s_k-\lambda_0-(t_k\text{-}t_0)\dot{\lambda}\end{aligned} \tag{45}$$

According to Equation (22), the functional relationship between the frame transformation parameters is expressed in matrix form:

$$\begin{aligned}\tilde{\mathbf{z}}_k &= \begin{bmatrix}\tilde{\boldsymbol{\theta}}_k\\ \tilde{\mathbf{d}}_k\\ \tilde{s}_k\end{bmatrix}=\begin{bmatrix}\boldsymbol{\theta}_k-\boldsymbol{\psi}_0-(t_k-t_0)\,\dot{\boldsymbol{\psi}}\\ \mathbf{d}_k-\mathbf{g}_0-(t_k-t_0)\,\dot{\mathbf{g}}\\ s_k-\lambda_0-(t_k-t_0)\,\dot{\lambda}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\theta}_k\\ \mathbf{d}_k\\ s_k\end{bmatrix}+\\ &\begin{bmatrix}-\mathbf{I} & \mathbf{0} & \mathbf{0} & -(t_k-t_0)\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & -(t_k-t_0)\mathbf{I} & \mathbf{0}\\ 0 & 0 & -1 & 0 & 0 & -(t_k-t_0)\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}_0\\ \mathbf{g}_0\\ \lambda_0\\ \dot{\boldsymbol{\psi}}\\ \dot{\mathbf{g}}\\ \dot{\lambda}\end{bmatrix}\equiv\mathbf{z}_k+\mathbf{E}_{\mathbf{z}_i}\mathbf{p}\end{aligned} \tag{46}$$

By applying Equation (41) to combine each station and Equation (46) to combine each observation epoch t_k, the inner constraints matrix for the intra-technique combination is obtained as follows:

$$\begin{aligned}\begin{bmatrix}\tilde{a}\\ \tilde{\mathrm{z}}\end{bmatrix} &= \begin{bmatrix}\tilde{a}_1\\ \vdots\\ \tilde{a}_N\\ -\\ z_1\\ \vdots\\ z_M\end{bmatrix}=\begin{bmatrix}a_1+E_{a_i}p\\ \vdots\\ a_N+E_{a_N}p\\ -\\ z_1+E_{z_i}p\\ \vdots\\ z_M+E_{z_M}p\end{bmatrix}=\begin{bmatrix}a_1\\ \vdots\\ a_N\\ -\\ z_1\\ \vdots\\ z_M\end{bmatrix}+\begin{bmatrix}E_{a_i}\\ \vdots\\ E_{a_N}\\ -\\ E_{z_i}\\ \vdots\\ E_{z_M}\end{bmatrix}p\\ &= \begin{bmatrix}a\\ \mathrm{z}\end{bmatrix}+\begin{bmatrix}E_a\\ E_z\end{bmatrix}p=\begin{bmatrix}a\\ \mathrm{z}\end{bmatrix}+Ep\end{aligned} \tag{47}$$

The inner constraints are expressed as:

$$\begin{aligned}0 &= E^T\begin{bmatrix}a\\ \mathrm{z}\end{bmatrix}=\begin{bmatrix}E_a^T & E_z^T\end{bmatrix}\begin{bmatrix}a\\ \mathrm{z}\end{bmatrix}=E_a^T a+E_z^T\mathrm{z}\\ &= \begin{bmatrix}E_{a_i}^T & \cdots & E_{aN}^T\end{bmatrix}\begin{bmatrix}a_1\\ \vdots\\ a_N\end{bmatrix}+\begin{bmatrix}E_{z_i}^T & \cdots & E_{z_N}^T\end{bmatrix}\begin{bmatrix}z_1\\ \vdots\\ z_M\end{bmatrix}\\ &= \sum_{i=1}^{N}E_{a_i}^T a_i+\sum_{i=1}^{M}E_{z_k}^T z_k\\ &= \begin{bmatrix}-\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}\boldsymbol{\theta}_k\\ \sum_{i=1}^{N}\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}\mathbf{d}_k\\ \sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}s_k\\ -\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)\boldsymbol{\theta}_k\\ \sum_{i=1}^{N}\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)\mathbf{d}_k\\ \sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)s_k\end{bmatrix}=\mathbf{0}\end{aligned} \tag{48}$$

$$\mathbf{E}_{\mathbf{a}_i}=\begin{bmatrix}[\mathbf{x}_{0i}^{\mathrm{ap}}\times] & \mathbf{I} & \mathbf{x}_{0i}^{\mathrm{ap}} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [\mathbf{x}_{0i}^{\mathrm{ap}}\times] & \mathbf{I} & \mathbf{x}_{0i}^{\mathrm{ap}}\end{bmatrix} \tag{49}$$

$$\mathbf{E}_{\mathbf{z}_k}=\begin{bmatrix}-\mathbf{I} & \mathbf{0} & \mathbf{0} & -(t_k-t_0)\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & -(t_k-t_0)\mathbf{I} & \mathbf{0}\\ 0 & 0 & -1 & 0 & 0 & -(t_k-t_0)\end{bmatrix} \tag{50}$$

$$E_{\mathrm{a}}^T a=\begin{bmatrix}0 & [\mathbf{x}_{0i}^{\mathrm{ap}}\times]\\ 0 & I\\ 0 & \mathbf{x}_{0i}^{\mathrm{ap}}\\ [\mathbf{x}_{0i}^{\mathrm{ap}}\times] & 0\\ I & 0\\ \mathbf{x}_{0i}^{\mathrm{ap}} & 0\end{bmatrix}\begin{bmatrix}\delta\mathbf{x}_{0i}\\ \delta\mathbf{v}_i\end{bmatrix}=\begin{bmatrix}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{v}_i\\ \delta\mathbf{v}_i\\ (\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{v}\\ [\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{x}_{0i}\\ \delta\mathbf{x}_{0i}\\ (\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{x}_{0i}\end{bmatrix} \tag{51}$$

$$E_z^T z=\begin{bmatrix}0 & 0 & -I\\ 0 & -I & 0\\ -I & 0 & 0\\ 0 & 0 & \text{-}(\mathrm{t_k}\text{ - }\mathrm{t_0})\mathrm{I}\\ 0 & \text{-}(\mathrm{t_k}\text{ - }\mathrm{t_0})\mathrm{I} & 0\\ \text{-}(\mathrm{t_k}\text{ - }\mathrm{t_0})\mathrm{I} & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\theta}_k\\ \mathbf{d}_k\\ s_k\end{bmatrix}=\begin{bmatrix}-s_k\\ -\mathbf{d}_k\\ -\boldsymbol{\theta}_k\\ -(t_k-t_0)s_k\\ -(t_k-t_0)\mathbf{d}_k\\ -(t_k-t_0)\boldsymbol{\theta}_k\end{bmatrix} \tag{52}$$

The inner constraints above can be divided into six parts:
Three rotation inner constraints

$$\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{x}_{0i}+\sum_{k=1}^{M}\boldsymbol{\theta}_k=0 \tag{53}$$

Three translation inner constraints

$$\sum_{i=1}^{N}\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}\mathbf{d}_k=0 \tag{54}$$

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Three scale inner constraints

$$\sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}s_k=0 \tag{55}$$

Three rotation rate inner constraints

$$-\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)\boldsymbol{\theta}_k=0 \tag{56}$$

Three translation rate inner constraints

$$\sum_{i=1}^{N}\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)\mathbf{d}_k=0 \tag{57}$$

Three scale rate inner constraints

$$\sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)s_k=0 \tag{58}$$

The full set of inner constraints is equivalent to the minimum norm solution.

$$\begin{aligned}\left\|\begin{bmatrix}a\\z\end{bmatrix}\right\|^2&=\begin{bmatrix}a\\z\end{bmatrix}^T\begin{bmatrix}a\\z\end{bmatrix}=a^Ta+z^Tz\\&=\sum_i(\delta\mathrm{x}_{i0}^T\delta\mathrm{x}_{i0}+\delta v_{i0}^T\delta v_{i0})\\&+\sum_k(\boldsymbol{\theta}_k^T\boldsymbol{\theta}_k+\mathbf{d}_k^T\mathbf{d}_k+s_k^2)=\min\end{aligned} \tag{59}$$

Two sets of partial inner constraints can be derived separately from the above equations.Considering only the partial inner constraints for coordinates and velocities, $E_a^Ta=\sum_{i=1}^{N}E_{ai}^Ta_i=0$, satisfying the condition $\sum_i(\delta\mathrm{x}_{i0}^T\delta\mathrm{x}_{i0}+\delta v_{i0}^T\delta v_{i0})=\min$, which takes the following form:

$$\begin{aligned}&\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{x}_{0i}=0,\sum_{i=1}^{N}\delta\mathbf{x}_{0i}=0,\sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{x}_{0i}=0,\\&\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{v}_i=0,\sum_{i=1}^{N}\delta\mathbf{v}_i=0,\sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{v}_i=0.\end{aligned} \tag{60}$$

The first three constraint conditions in Formula (60) apply to observation networks of rigid blocks, while the second column applies to observation networks with linear time-varying deformation.
Considering only the partial inner constraints for the transformation parameters, $E_z^Tz=\sum_{k=1}^{M}E_{z_k}^Tz_k=0$ satisfying the condition $\sum_k(\theta_k^T\theta_k+d_k^Td_k+s_k^2)=\min$, which takes the following form:

$$\begin{aligned}&\sum_{k=1}^{M}\boldsymbol{\theta}_k=0,\sum_{k=1}^{M}\mathbf{d}_k=0,\sum_{k=1}^{M}s_k=0,\\&\sum_{k=1}^{M}(t_k-t_0)\boldsymbol{\theta}_k=0,\sum_{k=1}^{M}(t_k-t_0)\mathbf{d}_k=0,\sum_{k=1}^{M}(t_k-t_0)s_k=0.\end{aligned} \tag{61}$$

We have established the most complete set of Inner Constraints Algorithms for rank-deficient free networks caused by unknown parameters (scale, translation, rotation). In some cases, if scale or translation parameters are effectively estimated from valid data (for example, when establishing a geocentric coordinate frame using SLR), the corresponding inner constraints need to be removed.

4.2 Inner Constraints Model for Inter-Technique Combination

The inner constraints are:

$$\begin{aligned}0&=E^T\begin{bmatrix}a\\\mathrm{z}\end{bmatrix}=\begin{bmatrix}E_a^T & E_z^T\end{bmatrix}\begin{bmatrix}a\\\mathrm{z}\end{bmatrix}=E_a^Ta+E_z^T\mathrm{z}\\&=\sum_{i=1}^{N}E_{a_i}^Ta_i+\sum_{i=1}^{M}E_{z_T}^Tz_T=\begin{bmatrix}-\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}\boldsymbol{\theta}_{T_0}\\\sum_{i=1}^{N}\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}\mathbf{d}_{T_0}\\\sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{x}_{0i}-\sum_{k=1}^{M}s_{T_0}\\-\sum_{i=1}^{N}[\mathbf{x}_{0i}^{\mathrm{ap}}\times]\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)\boldsymbol{\theta}_T\\\sum_{i=1}^{N}\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)d_T\\\sum_{i=1}^{N}(\mathbf{x}_{0i}^{\mathrm{ap}})^T\delta\mathbf{v}_i-\sum_{k=1}^{M}(t_k-t_0)s_T\end{bmatrix}\\&=\mathbf{0}\end{aligned} \tag{62}$$

The full set of inner constraints is equivalent to the minimum norm solution.

$$\begin{aligned}\left\|\begin{bmatrix}x\\z\end{bmatrix}\right\|^2&=\begin{bmatrix}x\\z\end{bmatrix}^T\begin{bmatrix}x\\z\end{bmatrix}=x^Tx+z^Tz\\&=\sum_i(\delta\mathrm{x}_{i0}^T\delta\mathrm{x}_{i0}+\delta v_i^T\delta v_i)\\&+\sum_{T}(\boldsymbol{\theta}_{T0}^T\boldsymbol{\theta}_{T0}+\mathbf{d}_{T0}^T\mathbf{d}_{T0}+s_{T0}^2+\dot{\boldsymbol{\theta}}_T^T\dot{\boldsymbol{\theta}}_T+\dot{\mathbf{d}}_T^T\dot{\mathbf{d}}_T+\dot{s}_T^2)\\&=\min\end{aligned} \tag{63}$$

5. Comparison of the Two-Step Method and the One-Step Method for Implementing Coordinate Reference Frames

The one-step method employs a synchronous combination algorithm: it first constructs the normal equations from the time series data of different observation techniques separately, then synchronously combines these technique-specific normal equations to form an overall normal equation. The optimal solution for the ITRF parameters is obtained by solving this overall normal equation (Jiang, et al., 2018).

The two-step method for establishing a global terrestrial reference framework，in the first step, using an intra-technique combination algorithm to separately solve the normal equations composed of time series data from different techniques, estimating the initial values of the ITRF parameters. In the second step, an inter-technique combination algorithm is employed to jointly construct and solve the normal equations using the initial values of the ITRF parameters obtained in the first step, thereby deriving the optimal solution of the ITRF parameters.

The preceding sections have researched and analyzed the fundamental theory and implementation method for the renewal of the global CGCS2000 coordinate reference framework using the two-step method model. The adoption of the two-step method model for realizing the globalized CGCS2000 reference framework is recommended for the following reasons: 1) Currently, the international IGS organization has not incorporated BeiDou data into the integrated adjustment for ITRF determination, and the latest released ITRF does not include BeiDou stations as fundamental stations. 2) The currently published CGCS2000 framework results do not utilize globally distributed VLBI, DORIS, and SLR data; the data processing stage only uses IGS data from domestic and surrounding stations. 3) BeiDou station data were not used during the initial establishment of the CGCS2000 coordinate framework, and furthermore, BeiDou receiver stations currently lack a globally uniform distribution.

Adopting the two-step method model also offers the following advantages: 1) It allows for the separate estimation of global CGCS2000 frame parameters from the results of different space techniques. Comparative analysis can reveal inter-technique systematic biases, such as scale bias, accuracy differences, and observational outliers. 2) Variance component estimation can be performed on the frame parameters from different space techniques obtained in the first step, enabling independent weight determination. Utilizing co-located site observation results and applying inter-technique combination processing methods allows for the scientific estimation of global CGCS2000 coordinate frame information that includes BeiDou observation station data. 3) Since the previously described regional CGCS2000 frame processing has already incorporated global IGS results, the loosely constrained solutions of the regional CGCS2000 frame stations from prior computations can be directly integrated into the two-step method model for processing. Through combined adjustment with other observation results, the final global CGCS2000 coordinate reference frame results can be obtained (Jiang, et al., 2018).

6. Conclusion

Based on the current status of China’s regional CGCS2000 coordinate reference framework and the urgent demand for global services from the third-generation BeiDou Navigation Satellite System (BDS-3), this paper establishes a mathematical model for the construction of a global CGCS2000 coordinate reference framework, develops a fusion processing algorithm integrating multiple space geodetic techniques, and resolves the issue of effective connection between the regional CGCS2000 framework and the international terrestrial reference frame. A two-step method is proposed for the realization of the global CGCS2000 reference framework, which offers the advantage of estimating ITRF parameters separately from the solutions of different space techniques and assigning appropriate weights to the frame parameters derived from various techniques via variance component estimation. The inner constraints theory is investigated, and the constraint conditions for coordinate reference are redefined based on the minimization of frame transformation parameters and their rates of change. As a result, the adjusted network achieves the highest degree of fit with the initial network geometry while preserving the intrinsic purity of reference frameworks established from different observation techniques. The findings of this study contribute to the refinement of the fundamental theoretical system for China’s coordinate reference framework establishment and provide scientific methodologies for the globalization of CGCS2000.

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References

Altamimi, Z., et al., 2007. ITRF2005: A new release of the International Terrestrial Reference Frame based on time series of station positions and Earth Orientation Parameters., *Journal of Geophysical Research: Solid Earth*., 112(B9).

Blewitt, G., Lavallée, D., 2002. Effect of annual signals on geodetic velocity., *Journal of Geophysical Research: Solid Earth*., 107(B7)

Dermanis, A., 2014. On the alternative approaches to ITRF formulation., *Earth on the Edge: Science for a Sustainable Planet*. Springer, 223-229.

Golub, G. H., et al., 1979. Generalized cross-validation as a method for choosing a good ridge parameter., *Technometrics*., 21(2),215-223.

Jiang Zhihao, Liu Jingnan, Wang Fan, et al., 2018. Research on Construction Theory of Global CGCS2ooo Coordinate Frame., *Geomatics and Information Science of Wuhan University.*, 43(02),167-174.

Sillard, P., Boucher, C., 2001. A review of algebraic constraints in terrestrial reference frame datum definition., *Journal of Geodesy*., 75(2-3),63-73.

Williams, S. D., Willis, P., 2006. Error analysis of weekly station coordinates in the DORIS network., *Journal of Geodesy*., 80(8/11),525-539.